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CYCLIC PROGRESSIVE NUMBER SYSTEMS

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1. *Introduction.* In certain computer applications to engineering problems such as the automatic control of machine tools the computing equipment derives its input in digital form, from, for example, an angle-indicating device on a rotating shaft. The indicating device may take the form of a multiple commutator or a coded disc read by photoelectric means. Since the shaft is rotating at mechanical speeds and the reading device is operating at electronic speeds a possibility of false readings exists when for example a change from 1999 to 2000 is taking place. In fact any of the readings 1999, 1990, 1900, 1000 or 2000 may occur in the same sort of way that a mileage recorder could be misread under similar circumstances.

This sort of difficulty can be avoided by making use of a special scale of numbering so arranged that an increase of one unit in the number represented requires a change in one and only one of the digits of the corresponding code. A code with this property is called a cyclic progressive code and a simple decimal example is as follows:

(*a*) 0, 1, 2, 3, ..., 8, 9, 10, 11, 12, ..., 18, 19, 20, 21, ...,
(*b*) 0, 1, 2, 3, ..., 8, 9, 19, 18, 17, ..., 11, 10, 20, 21, ...,
(*a*) 29, 30, 31, ..., 98, 99, 100, 101, ..., 109, 110, 111, ...,
(*b*) 29, 39, 38, ..., 91, 90, 190, 191, ..., 199, 189, 188, ...,

where rows (*a*) are ordinary decimals in normal sequence and rows (*b*) are their equivalents in a cyclic progressive decimal code.

Any two adjacent numbers in row (*b*) differ by one in exactly one digit and so sample readings from the encoder can never be in error by more than one unit. The representation above defines a one–one correspondence between pure decimals and these cyclic progressive decimals, but there are in fact other equally valid cyclic progressive representations. Furthermore, cyclic progressive number systems occur to any number base. In the next paragraph rules for conversion from pure to a form of cyclic progressive numbers and vice versa are formulated.

From an engineering point of view the problem of addition and multiplication of numbers in cyclic progressive form does not arise since the computer is operating at electronic speeds and it is therefore quite practicable to make a conversion to the pure number base, carry out the necessary arithmetic operations and finally to convert back to cyclic progressive form to feed back instructions to the tool. From a mathematical point of view the problem is of interest and

this is the justification for the remainder of this paper. In particular one interesting property of cyclic progressive numbers is that the digits do not have a basic unique interpretation as do the digits of a pure decimal. For example, in the above system the cyclic progressive number 1929 is equivalent to the pure decimal 1029 and clearly the two nines in the first number are interpreted differently.

Little appears in print on the theory of cyclic progressive systems. E. J. Petherick [1] describes a cyclic progressive binary-coded system and mentions some results of E. E. Wright on the conversion of even based numbers to a cyclic progressive form. A. J. Cole [2] also makes a brief reference to such systems.

2. *Conversion rules.* A general rule for conversion from a pure number system to any even base b to a cyclic progressive system may be stated as follows. A digit p_n in the normal sequence remains unaltered if its more significant (that is, left hand) neighbour is even and is replaced by its complement if its more significant neighbour is odd where the complement of a digit p_n to base b is defined as

$$b-1-p_n.$$

Thus in a decimal code the complement of p_n is $9-p_n$ and the normal decimal number 27435 becomes the cyclic progressive number 27534. Similarly in a pure binary code the complement of p_n is $1-p_n$ and the pure binary number 110010 becomes the cyclic progressive number 101011.

It is easy to see that this conversion does generate a cyclic progressive system for if the least significant digit of a pure number to base b is not $b-1$ then the addition of 1 will only change the least significant digit and correspondingly the least significant digit of the cyclic progressive number will increase or decrease by 1 depending on the parity of the next most significant digit. On the other hand if the pure number finishes with a string of n digits $b-1$ where $n \geqslant 1$ then the addition of 1 changes all these digits to zero and also changes the parity of the next more significant digit. Now since $b-1$ is odd and 0 is even, the $n-2$ least significant digits of both cyclic progressive numbers will be zero and so it suffices to consider adjacent pairs in the pure number of the form

$$\text{odd} \quad b-1$$

or

$$\text{even} \quad b-1$$

where the odd number in the first case is not $b-1$. The addition of 1 to these pure numbers converts them to

$$\text{even} \quad 0$$

and

$$\text{odd} \quad 0$$

In either case the corresponding cyclic progressive numbers change by one in the first digit only. Table 1 lists some typical examples of adjacent pure and cyclic decimals.

TABLE 1. *Examples of decimal conversion*

Pure decimal	Cyclic Progressive
319999	380000
320000	370000
329999	379000
330000	369000
459999	450000
460000	460000
469999	469000
470000	479000

This proof breaks down when the number system has an odd base b since in this case 0 and $b-1$ are both even and indeed this invalidates the above conversion rule. For example 2122 and 2200 are consecutive integers to base 3 and these would convert to 2102 and 2200 respectively and the converted system is certainly not cyclic progressive.

A rule for converting pure numbers to an odd base b to a cyclic progressive form is as follows. A digit p_n in the normal sequence remains unaltered if the sum of all its more significant digits is even and is replaced by its complement $b-1-p_n$ if this sum is odd. For example to base 3 the consecutive pure numbers 2122 and 2200 become cyclic progressive numbers 2100 and 2200. The proof that this conversion does in fact generate a cyclic progressive system is similar to that for an even base conversion but is rather more wordy. Table 2 lists some critical cases for numbers to base 7.

The corresponding conversion from a cyclic progressive number

TABLE 2. *Conversion to base* 7

Pure number to base 7	Cyclic progressive
116666	156666
120000	146666
126666	140000
130000	130000
226666	226666
230000	236666
236666	230000
240000	240000

derived by either of the above rules back to a pure number to base b can be covered by a single rule as follows. A digit c_n remains unaltered if the sum of all its more significant neighbours is even; otherwise it is replaced by its complement $b-1-c_n$.

3. *Additions and Subtraction.* The digit s_n arising from the sum of two cyclic progressive digits a_n and b_n depends on all the more significant digits of a_n and b_n. To fix ideas we consider a cyclic progressive decimal system. The digits are then given an upper prefix 0 or ′ depending on the parity of the sum of all the more significant digits, namely 0 if this sum is even and ′ if it is odd. The cyclic progressive decimal

$$3281477$$

would then be written

$$^{0}3\ {}'2\ {}'8\ {}'1\ {}^{0}4\ {}^{0}7\ {}'7.$$

The addition table is more complicated than the pure decimal one since the sum of digits ${}^{i}a$, ${}^{j}b$ where i, j take all combinations of 0 and ′ must be accommodated. Table 3 shows such an addition table where the table lookup occurs from the appropriately chosen digits on the whole perimeter and where if the upper prefixes of the augend and addend are mixed then the table entry, ignoring carry, must be *complemented on* 9.

TABLE 3. *Cyclic Progressive Decimal Addition*

	′0	′1	′2	′3	′4	′5	′6	′7	′8	′9	
′0	$_1 8$	$_1 7$	$_1 6$	$_1 5$	$_1 4$	$_1 3$	$_1 2$	$_1 1$	$_1 0$	9	$^0 9$
′1	$_1 7$	$_1 6$	$_1 5$	$_1 4$	$_1 3$	$_1 2$	$_1 1$	$_1 0$	9	8	$^0 8$
′2	$_1 6$	$_1 5$	$_1 4$	$_1 3$	$_1 2$	$_1 1$	$_1 0$	9	8	7	$^0 7$
′3	$_1 5$	$_1 4$	$_1 3$	$_1 2$	$_1 1$	$_1 0$	9	8	7	6	$^0 6$
′4	$_1 4$	$_1 3$	$_1 2$	$_1 1$	$_1 0$	9	8	7	6	5	$^0 5$
′5	$_1 3$	$_1 2$	$_1 1$	$_1 0$	9	8	7	6	5	4	$^0 4$
′6	$_1 2$	$_1 1$	$_1 0$	9	8	7	6	5	4	3	$^0 3$
′7	$_1 1$	$_1 0$	9	8	7	6	5	4	3	2	$^0 2$
′8	$_1 0$	9	8	7	6	5	4	3	2	1	$^0 1$
′9	9	8	7	6	5	4	3	2	1	0	$^0 0$
	$^0 9$	$^0 8$	$^0 7$	$^0 6$	$^0 5$	$^0 4$	$^0 3$	$^0 2$	$^0 1$	$^0 0$	

A lower prefix in the body of the table denotes carry. No upper suffix can be included since this depends on the surrounding digits of both augend and addend. Table 4 gives some examples of the sum of single digits.

TABLE 4. *Sum of single cyclic progressive digits*

${}'3 + {}'5 = {}_{1}0$
${}^{0}2 + {}^{0}7 = 9$
${}^{0}4 + {}'5 = 9 - 8 = 1$
${}'6 + {}^{0}8 = 9 - {}_{1}1 = {}_{1}8$

The sum of two cyclic progressive numbers which do not produce carry is now obvious. For example, the pure decimal sum

$$\begin{array}{c} 14430 \\ 33229 \\ \hline 47659 \end{array}$$

corresponds to the cyclic progressive sum

$$\begin{array}{c} {}^{0}1\ {}'5\ {}^{0}4\ {}^{0}3\ {}'9 \\ {}^{0}3\ {}'6\ {}'7\ {}^{0}2\ {}^{0}9 \\ \hline 4\ \ 7\ \ 3\ \ 5\ \ 0 \end{array}$$

and the two answers check by the conversion law.

Now consider the decimal addition

$$\begin{array}{c} 39147 \\ 27638 \\ \hline 66785 \end{array}$$

which involves carry. The first stage of the corresponding cyclic progressive form gives

$$\begin{array}{c} {}^{0}3\ {}'0\ {}'8\ {}'5\ {}^{0}7 \\ {}^{0}2\ {}^{0}7\ {}'3\ {}^{0}3\ {}'1 \\ \hline 5\ {}_{1}3\ \ 7\ \ 2\ {}_{1}4 \end{array}$$

The carry in pure decimal form is

$$10010$$

or, in cyclic progressive form

$$19019$$

and this can be added in as usual

$$\begin{array}{c} {}^{0}5\ {}'3\ {}^{0}7\ {}'2\ {}'4 \\ {}^{0}1\ {}'9\ {}^{0}0\ {}^{0}1\ {}'9 \\ \hline 6\ \ 6\ \ 7\ \ 1\ \ 5 \end{array}$$

which checks with the pure decimal result.

This method is unsatisfactory since it uses the pure decimal form of the carry. This can be avoided as follows. Treat all the carry digits as having prefix 0. Then, working from the right, corresponding to each carry digit complement the digit to which it is attached on 9 and add the carry digit itself into the next digit by Table 3. Thus in the above example

$${}^{0}5\ {}_{1}^{\prime}3\ {}^{0}7\ {}'2\ {}_{1}^{\prime}4$$

becomes

$$6\ \ 6\ \ 7\ \ 1\ \ 5.$$

If the addition of carry itself induces carry then this can be treated on the next line in the same way. For example, the pure decimal sum

$$\begin{array}{c} 35471 \\ 68563 \\ \hline 104034 \end{array}$$

becomes the cyclic progressive sum

$$\begin{array}{c} {}^{0}3\ {}'4\ {}'5\ {}^{0}7\ {}'8 \\ {}^{0}6\ {}^{0}8\ {}^{0}5\ {}'3\ {}^{0}3 \\ \hline {}^{0}9\ {}_{1}^{\prime}6\ {}'0\ {}_{1}^{\prime}6\ {}'5 \\ \hline {}_{1}^{0}0\ {}^{0}3\ {}_{1}^{\prime}9\ {}^{0}3\ {}'5 \\ \hline 1\ 0\ \ 4\ \ 0\ \ 3\ \ 5 \end{array}$$

Subtraction can be performed by a method equivalent to adding a complement in the pure decimal case. First make up both numbers to the same length. The complement of a cyclic progressive number is then found by complementing its most significant digit on 9 even when this most significant digit is 0. If on addition of this complement there is carry past the most significant digit the answer is positive and the carry digit should be dropped and *added* into the least significant end of the answer, rather like " end about carry " in some computer addition. If there is no carry the answer is negative in complementary form and to find its numerical value complement the most significant digit on 9 unless it is itself 9 in which case replace any string of 9's at the most significant end by 0's. Thus the two pure decimal differences

$$\begin{array}{r} 3\;8\;0\;2\;6\;3 \\ 1\;7\;4\;6\;3\;7 \\ \hline 2\;0\;5\;6\;2\;6 \end{array} \qquad \begin{array}{r} 1\;7\;4\;6\;3\;7 \\ 3\;8\;0\;2\;6\;3 \\ \hline -2\;0\;5\;6\;2\;6 \end{array}$$

correspond to the cyclic progressive sums

$$\begin{array}{r} {}^{0}3\;{}'1\;{}^{0}0\;{}^{0}2\;{}^{0}6\;{}^{0}3 \\ {}^{0}8\;{}^{0}2\;{}^{0}5\;{}'6\;{}'3\;{}^{0}2 \\ \hline {}^{0}_{1}1\;{}'_{1}9\;{}^{0}5\;{}'4\;{}'_{1}7\;{}^{0}5 \\ \hline 1\;{}^{0}2\;{}^{0}0\;{}^{0}5\;{}'3\;{}^{0}2\;{}^{0}5 \\ \hline +\quad 2\;0\;5\;3\;2\;6 \end{array} \qquad \begin{array}{r} {}^{0}1\;{}'2\;{}'5\;{}^{0}6\;{}^{0}3\;{}'2 \\ {}^{0}6\;{}^{0}1\;{}'0\;{}'2\;{}'6\;{}'3 \\ \hline {}^{0}7\;{}'1\;{}^{0}_{1}3\;{}'_{1}6\;{}'3\;{}^{0}_{1}3 \\ \hline 7\;0\;5\;3\;2\;6 \\ \hline -2\;0\;5\;3\;2\;6 \end{array}$$

which again check with the decimal form.

It is possible to simplify the addition of carry by defining a larger addition table. For a system to base b this requires a $2b \times 2b$ table. For example, the corresponding binary addition table is shown in Table 5. Note that the upper prefixes are now defined by the table position.

TABLE 5. *Extended cyclic progressive binary addition table*

	${}^{0}0$	${}^{0}1$	${}'1$	${}'0$
${}^{0}0$	${}^{0}0$	${}^{0}1$	${}'1$	${}'0$
${}^{0}1$	${}^{0}1$	${}'_{1}1$	${}'0$	${}^{0}_{1}0$
${}'1$	${}'1$	${}'0$	${}^{0}0$	${}^{0}1$
${}'0$	${}'0$	${}^{0}_{1}0$	${}^{0}1$	${}'_{1}1$

The addition of carry now no longer requires the digit to which the carry is attached to be complemented.

For example the pure binary examples

$$\begin{array}{r} 1\;0\;1\;1\;0\;0\;1 \\ 1\;1\;0\;1\;1\;0\;1 \\ \hline 1\;1\;0\;0\;0\;1\;1\;0 \end{array} \qquad \begin{array}{r} 1\;1\;0\;1\;1\;0\;1 \\ -1\;0\;1\;1\;0\;1\;1 \\ \hline 0\;0\;1\;0\;0\;1\;0 \end{array} \qquad \begin{array}{r} 1\;0\;1\;1\;0\;1\;1 \\ -1\;1\;0\;1\;1\;0\;1 \\ \hline -0\;0\;1\;0\;0\;1\;0 \end{array}$$

correspond to the cyclic progressive sums

$$\begin{array}{r} {}^{0}1\;{}'1\;{}^{0}1\;{}'0\;{}'1\;{}^{0}0\;{}^{0}1 \\ {}^{0}1\;{}'0\;{}'1\;{}^{0}1\;{}'0\;{}'1\;{}^{0}1 \\ \hline 1\;0\;1\;0\;0\;1\;0\;1 \end{array} \qquad \begin{array}{r} {}^{0}1\;{}'0\;{}'1\;{}^{0}1\;{}'0\;{}'1\;{}^{0}1 \\ {}^{0}0\;{}^{0}1\;{}'1\;{}^{0}0\;{}^{0}1\;{}'1\;{}^{0}0 \\ \hline 1\;{}'1\;{}^{0}0\;{}^{0}1\;{}'1\;{}^{0}0\;{}^{0}0\;{}^{0}1 \\ \hline +\;0\;0\;1\;1\;0\;1\;1 \end{array} \qquad \begin{array}{r} {}^{0}1\;{}'1\;{}^{0}1\;{}'0\;{}'1\;{}^{0}1\;{}'0 \\ {}^{0}0\;{}^{0}0\;{}^{0}1\;{}'1\;{}^{0}0\;{}^{0}1\;{}'1 \\ \hline {}^{0}1\;{}'0\;{}'1\;{}^{0}1\;{}'0\;{}'1\;{}^{0}1 \\ \hline -\;0\;0\;1\;1\;0\;1\;1 \end{array}$$

where the carry has been added in mentally using the table, and " end about carry " has been applied in the second case but with the new condition that the most significant digit of the result is complemented on 1.

To conclude this section we give an example of an addition table to an odd base. Table 6 gives the cyclic progressive addition table to base 3.

TABLE 6. *Extended cyclic progressive addition table to base* 3

	${}^{0}0$	${}^{0}1$	${}^{0}2$	${}'2$	${}'1$	${}'0$
${}^{0}0$	${}^{0}0$	${}^{0}1$	${}^{0}2$	${}'2$	${}'1$	${}'0$
${}^{0}1$	${}^{0}1$	${}^{0}2$	${}'_{1}2$	${}'1$	${}'0$	${}^{0}_{1}0$
${}^{0}2$	${}^{0}2$	${}'_{1}2$	${}'_{1}1$	${}'0$	${}^{0}_{1}0$	${}^{0}_{1}1$
${}'2$	${}'2$	${}'1$	${}'0$	${}^{0}0$	${}^{0}1$	${}^{0}2$
${}'1$	${}'1$	${}'0$	${}^{0}_{1}0$	${}^{0}1$	${}^{0}2$	${}'_{1}2$
${}'0$	${}'0$	${}^{0}_{1}0$	${}^{0}_{1}1$	${}^{0}2$	${}'_{1}2$	${}'_{1}1$

In the following example normal addition to base 3 is given on the left and the corresponding cyclic progressive addition is given on the right.

$$\begin{array}{r} 1\;2\;1\;0\;2\;1 \\ 1\;0\;1\;2\;2\;2 \\ \hline 1\;0\;0\;0\;0\;2\;0 \end{array} \qquad \begin{array}{r} {}^{0}1\;{}'0\;{}'1\;{}^{0}0\;{}^{0}2\;{}^{0}1 \\ {}^{0}1\;{}'2\;{}'1\;{}^{0}2\;{}^{0}2\;{}^{0}2 \\ \hline 1\;2\;2\;2\;2\;0\;2 \end{array}$$

4. *Multiplication.* To illustrate cyclic progressive multiplication

we revert to binary numbers. Table 7 gives the cyclic progressive binary multiplication table in extended form.

TABLE 7. *Extended cyclic progressive binary multiplication table*

	$^{0}0$	$^{0}1$	$'1$	$'0$
$^{0}0$	$^{0}0$	$^{0}0$	$^{0}0$	$^{0}0$
$^{0}1$	$^{0}0$	$^{0}1$	$'1$	$'0$
$'1$	$^{0}0$	$'1$	${}^{0}_{'1}1$	${}^{\prime}_{'1}1$
$'0$	$^{0}0$	$'0$	${}^{\prime}_{'1}1$	${}^{0}_{'1}1$

In the following example the pure binary case is given on the left and the corresponding cyclic progressive case is given on the right. To simplify the printing the carry digits from cyclic progressive multiplication have been added into the next most significant digit on the same line. This makes the rows on the right the cyclic progressive equivalents of the pure binary on the left.

1 0 0 1 1 0	$^{0}1\ '1\ ^{0}0\ ^{0}1\ '0\ '1$
1 1 0 0 1 1	$^{0}1\ '0\ '1\ ^{0}0\ ^{0}1\ '0$
1 0 0 1 1 0	$^{0}1\ '1\ ^{0}0\ ^{0}1\ '0\ '1$
1 0 0 1 1 0	$^{0}1\ '1\ ^{0}0\ ^{0}1\ '0\ '1$
0 0 0 0 0 0	$^{0}0\ ^{0}0\ ^{0}0\ ^{0}0\ ^{0}0\ ^{0}0$
0 0 0 0 0 0	$^{0}0\ ^{0}0\ ^{0}0\ ^{0}0\ ^{0}0\ ^{0}0$
1 0 0 1 1 0	$^{0}1\ '1\ ^{0}0\ ^{0}1\ '0\ '1$
1 0 0 1 1 0	$^{0}1\ '1\ ^{0}0\ ^{0}1\ '0\ '1$
1 1 1 1 0 0 1 0 0 1 0	1 0 0 0 1 0 1 1 0 1 1

5. *Inequalities and division.* In order to be able to perform long division it is necessary to be able to compare the relative magnitude of two cyclic progressive numbers. As with pure numbers to the same base the digit which determines the relative magnitude of two cyclic progressive numbers is the first, starting from the left, which is not identical in the two numbers. Let these corresponding digits in the two numbers be a and b. The upper prefixes of these two digits must be identical because all the more significant digits are identical. The relative magnitude of the two numbers is then determined by the relative positions of a and b in the sequence

$$^{0}0\ ^{0}1\ '1\ '0.$$

Thus

$$1\,0\,1\,1\,0\,1 > 1\,0\,1\,1\,1\,1$$

since '0 is to the right of '1 whereas

$$1\,0\,1\,0\,0\,1 < 1\,0\,1\,1\,0\,1$$

since $^{0}0$ is to the left of $^{0}1$ in the above sequence.

Division can now be performed in a similar manner to ordinary long division by using this idea of an inequality together with multiplication and subtraction by the addition of a complement. To conclude we give a very simple example of division of the cyclic progressive binary number 10110 by 110. This corresponds to the pure binary division of 11011 by 101 giving the quotient 101 with remainder 10.

$1\ '1\ ^{0}1$	
$^{0}1\ '1\ ^{0}0\ )\ ^{0}1\ '0\ '1\ ^{0}1\ '0$	
$^{0}0\ ^{0}1\ '0\ '0\ '0$	Complement of 11000
$1\ '1\ ^{0}1\ '1\ ^{0}1\ '1$	End about carry to be applied.
$0\ ^{0}1\ '1\ ^{0}1\ '0$	
$^{0}1\ '0\ '0\ '0$	Complement of 0000
$1\ '0\ '1\ ^{0}0\ ^{0}1$	End about carry to be applied.
$1\ '1\ ^{0}1\ '1$	
$^{0}0\ ^{0}1\ '0\ '0$	Complement of 1100
$1\ '1\ ^{0}0\ ^{0}0\ ^{0}1$	End about carry to be applied.
0 0 1 1	Remainder.

REFERENCES

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