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ON A PROBLEM IN PARITY

By

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1. I have recently proved that

$$v_m(n)=(n+m-1)!\,(n, m)/n!\,m!,$$

is odd exactly

$$Q(w)=(3^{w+1}-2^{w+2}+2w+1)/4$$

times, as m runs through positive integers $\leqslant n$, while n takes all positive integral values $\leqslant 2^w-1$, $w \geqslant 1$.

Here, we give an alternative proof of this result, we also consider the case when $(n, m)=1$.

Throughout this note

(i) $1 \leqslant m \leqslant n$:

(ii) A value of the number-pair $\{n; m\}$ is said to be acceptable if it makes $v_m(n)$ odd;

(iii) All congruences are modulo 2;

(iv) If $h(\geqslant 0)$ be the highest power of 2 which divides u, then we write

$$\text{pot } u=h;$$

(v) u^* stands for $[u/2]$.

Moreover, we write

$$u=(a_j\, a_{j-1} \ldots a_2 a_1 a_0)$$

in the binary scale. All numbers $<2^{j+1}$ can be expressed in this manner. If $2^j \leqslant u < 2^{j+1}$, $a_j=1$; if $2^j > u$, $a_j=0$.

While dealing with numbers less than 2^{j+1}, we sometimes express them uniformly with $(j+1)$ figures each, in the binary scale. Thus, each of the a's can be zero or 1. To distinguish between the two cases, the representation is said to be proper when a_j, that is the figure to the extreme left in the representation, is 1.

It is easy to show that for $u < 2^{j+1}$,

$$\text{pot}\,(u!) = u - \sum_{t=0}^{j} a_t = u - \Sigma a.$$

2. Our proof of the result stated in section 1, is based on the

Theorem. *If n and m are both even, and* pot n = pot m, *then*

$$v_m(n) \equiv \binom{n^* + m^* - 1}{m^* - 1}.$$

In all other cases,

$$v_m(n) \equiv \binom{n^* + m^*}{m^*}.$$

Proof. (i) When at least one of n and m is odd, we delete odd factors from the numerator and the denominator of $v_m(n)$ and divide each of the even factors by 2, and the result follows.

(ii) When n and m are both even, but

$$\text{pot}\, n \neq \text{pot}\, m,$$

we observe that

$$\text{pot}\,(n, m) = \text{pot}\,(n+m),$$

so that

$$v_m(n) \equiv (n+m)!/n!\,m!,$$

$$\equiv \binom{n^* + m^*}{m^*}.$$

(iii) Finally, when n and m are both even and pot n = pot m, we have

$$v_m(n) \equiv (n+m-1)!/n!(m-1)!,$$

$$\equiv (n+m-2)!/n!(m-2)!,$$

$$\equiv\binom{n^*+m^*-1}{m^*-1}.$$

We notice that when n and m are both even and pot n=pot m, then

$$\binom{n^*+m^*}{m^*}$$

is even.

3. Let $2^j \leqslant n < 2^{j+1}, j \geqslant 1$. Then, in the binary scale, we can write

$$n^*=(a_j \dots a_2 a_1 a_0),$$

$$m^*=(b_j \dots b_2 b_1 b_0),$$

and

$$n^*+m^*=(c_j \dots c_2 c_1 c_0).$$

Evidently, then

$$\binom{n^*+m^*}{m^*}$$

is odd, if and only if,

(3.1) $$\Sigma a+\Sigma b=\Sigma c.$$

This holds, if and only if,

(3.2) $$a_t+b_t=0 \text{ or } 1,\ t=0, 1, 2, \dots j.$$

Since $a_j=0=b_j$, we must have $c_j=0$ also.

Moreover, since $a_{j-1}=1$, (3.2) can hold only if $b_{j-1}=0$. Now (3.2) has three solutions for each value of t from 0 to $(j-2)$. This includes the one case when every b_t is zero. In this case, (3.2) has only two solutions for every t, $0 \leqslant t \leqslant j-2$.

Thus, if $m^* \geqslant 1$, the number-pair $\{n^*; m^*\}$ can have $(3^{j-1}-2^{j-1})$ sets of values, leading to $4(3^{j-1}-2^{j-1})$ acceptable values for the number-pair $\{n; m\}$. While if $m^*=0$, $\{n^*; m^*\}$ can have only 2^{j-1} values, to which correspond 2^j acceptable values of $\{n; m\}$. These conclusions follow from the fact that for each value of $u^* \geqslant 1$, u has two values one of which is odd and the other even; while for $u^*=0$, u has only one value. Actually, when $u^* \geqslant 1$, $u=2u^*$ or $2u^*+1$; while for $u^*=0$, $u=1$.

4. Now, consider the case when both n and m are even and

$$\text{pot } n = \text{pot } m.$$

Again, let $2^j \leqslant n < 2^{j+1}, j \geqslant 1$. Write

$$n^* = (a_j \ldots a_2 a_1 a_0),$$

$$m^* = (b_j \ldots b_2 b_1 b_0),$$

$$m^* - 1 = (d_j \ldots d_2 d_1 d_0),$$

$$n^* + m^* - 1 = (e_j \ldots e_2 e_1 e_0).$$

Then, $\binom{n^*+m^*-1}{m^*-1}$ is odd, if and only if,

(4.1) $$a_t + d_t = 0 \text{ or } 1, \; t = 0, 1, 2, \ldots j.$$

Since

$$a_{j-1} = 1, \text{ and } a_j = 0 = d_j \;;$$

in view of (4.1), we must have

$$d_{j-1} = 0, \text{ and } e_j = 0.$$

Now, let q be the least integer $\geqslant 0$, for which $a_q = 1$.

Then, since pot n = pot m, we must have

$$b_t = 0 \text{ for } t = 0, 1, 2, \ldots, q-1; \quad \text{and } b_q = 1.$$

Hence,

$$d_t = 1, \text{ for } t = 0, 1, 2, \ldots, q-1; \text{ and } d_q = 0.$$

For $q < j-2$, (4.1) has three solutions for each t from 0 to $(j-3)$ while if $q = (j-2)$ or $(j-1)$, (4.1) has just one solution in each of the two cases.

Since, n and m are both even, the number of acceptable number-pairs $\{n \; ; \; m\}$ in this case is

(4.2) $$2 + \sum_{q=0}^{j-3} 3^{j-2-q} = (3^{j-1} + 1)/2.$$

5. Making use of the results proved in the preceding two sections and noting that $v_1(1)$ is odd, we find that for

$$1 \leqslant m \leqslant n \leqslant 2^w - 1,$$

the expression for $v_m(n)$ is odd exactly

$$1 + \sum_{j=1}^{w-1} \{(4.3^{j-1} - 2^j) + ((3^{j-1} + 1)/2)\}$$

$$= (3^{w+1} - 2^{w+2} + 2w + 1)/4$$

times as stated.

6. One can ask: "How often is $v_m(n)$ odd as m runs through values not exceeding n and prime to it, while n takes all positive integral values upto $2^w - 1$?".

The question appears to be difficult to answer, but a probabilistic argument leads to an approximate formula which is well supported by the available numerical evidence.

The calculations were carried out in the following manner with $n \geqslant 2$.

Take $n = 68, 69$; so that we have $n^* = (100010)$.

The values of m for which $\binom{n^*+m^*}{m^*}$ is odd, are readily seen to be

1, 2, 3, 8, 9, 10, 11;
16, 17, 18, 19, 24, 25, 26, 27;
32, 33, 34, 35, 40, 41, 42, 43;
48, 49, 50, 51, 56, 57, 58, 59.

(Evidently, the total number of values of m for a given n, is given by $2^{h+1} - 1$, where h is the number of zeros in the proper binary representation of n^*.)

Of the above noted values of m, exactly 14 are prime to 68, and 21 prime to 69.

(These numbers are closely approximated by

$$\phi(n) \; (2^{h+1} - 1)/n,$$

as could be expected. Thus

$$\phi(68).31/68=14{\cdot}6 \text{ and } \phi(69).31/69=19{\cdot}8.)$$

The total number of times that

$$\binom{n^*+m^*}{m^*}$$

is odd for $1\leqslant m\leqslant n\leqslant 2^w-1$, is given by

$$M(w)=2(3^{w-1}-2^{w-1})+1.$$

Since, for large n,

$$\frac{\sum_{k=1}^{n}\phi(k)}{\sum_{k=1}^{n}k}\sim\frac{6}{\pi^2},$$

we can expect $v_m(n)$ to be odd, when $1\leqslant m\leqslant n\leqslant 2^w-1$, and $(n, m)=1$, about $6M(w)/\pi^2$ times. Actually

(6.1) $$N(w)=[6M(w)/\pi^2]+2^{w-1},$$

provides a very good approximation to the actual results for $w\leqslant 7$, as will be seen from the following table:

w	Actual count	$N(w)$	$M(w)$
1	1	1	1
2	3	3	3
3	10	10	11
4	29	31	39
5	97	95	131
6	284	289	423
7	871	873	1331

The counts having been made in two independent ways, can be fully relied upon.

The problem : "How often is $v_m(n)$ prime to an odd prime p, as m and n run through values for which $1\leqslant m\leqslant n\leqslant p^w-1$?", can be considered on the same lines.

Appendix

Table 1

giving for a given n, the number of m's such that $(m, n)=1$, $1\leqslant m\leqslant n$ and $v_m(n)$ is odd.

n	0	1	2	3	4	5	6	7	8	9
		1	1	1	2	3	1	1	4	5
1	1	3	1	3	1	1	8	15	3	7
2	4	5	2	3	3	6	2	3	2	3
3	1	1	16	19	7	10	5	15	4	5
4	7	15	3	7	4	5	2	3	5	13
5	3	5	4	7	1	3	3	5	2	3
6	1	3	1	1	32	47	11	31	14	21
7	6	15	11	31	7	9	7	12	3	7
8	12	21	7	15	4	11	4	5	7	15
9	2	7	4	5	2	3	11	31	7	11
10	7	15	2	7	8	8	4	7	3	7
11	2	3	7	15	3	6	4	5	2	3
12	2	7	2	3	2	3	1	1		

Table 2

giving for a given m, the number of n's for which $(n, m)=1$, $1\leqslant m\leqslant n\leqslant 127$ and $v_m(n)$ is odd.

m	0	1	2	3	4	5	6	7	8	9
		127	31	42	30	48	10	26	28	38
1	12	27	10	27	6	8	24	45	8	23
2	8	14	5	12	8	20	5	8	5	12
3	1	6	16	20	7	11	6	15	4	5
4	6	15	2	8	4	4	2	4	6	14
5	3	5	3	8	2	3	4	6	2	4
6	1	4	1	2	0	0	0	0	0	0
.	.	.	.	.	.	.	.	.	.	.
12	0	0	0	0	0	0	0	0		

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