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A GENERALIZATION OF THE PARTITION FUNCTION*

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1. We define $v_r(n)$ by the relation

$$nv_r(n) = \sum_{j=1}^{n} \sigma_r(j) v_r(n-j), \quad n \geq 1; \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

where $\sigma_r(j)$ denotes the sum of the rth powers of the divisors of j and $v_r(0) = 1$.

It would be readily seen that $v_1(n)$ is nothing but the function $p(n)$ denoting the number of unrestricted partitions of n.

We proceed to show that $v_r(n)$ is the coefficient of x^n in the expansion of

$$J(x) = \prod_{t=1}^{\infty} (1-x^t)^{-t^{r-1}}, \quad |x| < 1 . \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Let
$$J(x) = \sum_{t=0}^{\infty} u_t x^t, \quad u_0 = 1 .$$

Differentiating (2) logarithmically with respect to x, we obtain

$$\sum_{t=0}^{\infty} t u_t x^t \Big/ \sum_{t=0}^{\infty} u_t x^t = xJ'(x) / J(x).$$

$$= \left\{ \frac{1^r x}{1-x} + \frac{2^r x^2}{1-x^2} + \frac{3^r x^3}{1-x^3} + \cdots \right\}$$

$$= \sum_{j=1}^{\infty} \sigma_r(j) x^j$$

or
$$\sum_{t=0}^{\infty} t u_t x^t = \sum_{t=0}^{\infty} u_t x^t \cdot \sum_{j=1}^{\infty} \sigma_r(j) x^j \quad \ldots \quad (3)$$

Equating the coefficients of x^n on the two sides of (3), we get

$$n u_n = \sum_{j=1}^{n} \sigma_r(j) u_{n-j}, \quad n \geq j .$$

Since this relation is the same as (1), we must have

$$u_n = v_r(n).$$

From (2), it would be readily seen that $v_r(n)$ is the number of partitions of n when element k is of k^{r-1} different types, i.e. when there are 2^{r-1} different two's, 3^{r-1} different three's and so on.

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* Suggested by Dr. D. S. Kothari.

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If we now define $r_r(n, m)$ by the relation

$$nr_r(n, m) = \sum_{j=1}^{n} \sigma_r(j, m) r_r(n-j, m), \quad n \geq 1; \qquad \ldots \quad \ldots \quad (4)$$

where $\sigma_r(j, m)$ denotes the sum of the rth powers of those divisors of j which do not exceed m, then it can be shown that $r_r(n, m)$ is the coefficient of x^n in the expansion of

$$J(x, m) = \prod_{j=1}^{m} (1-x^j)^{-j^{r-1}}; \quad |x| < 1$$

Evidently, $r_r(n, m)$ denotes the number of partitions of n into parts not exceeding m, element k being of k^{r-1} different types.

Thus $\quad r_r(n, m) = r_r(n, n) = r_r(n), \quad m > n. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5)$

3. *Lemma.* If $(1-x)^{-1} \sum_{t=0}^{\infty} a_t x^t = \sum_{t=0}^{\infty} A_t x^t$,

and $\quad a_t = \binom{t+s}{s}$ for every $t \geq 0$, $0 \leq s < r$; then

$$\binom{t+s+1}{r+1} \leq jA_t \leq \binom{t+s+j}{r+1}$$

The proof is simple and is left to the reader.
This immediately leads us to the inequality (Gupta 1942)

$$\binom{n-1+S_{r-1}(m)}{S_{r-1}(m)-1} \leq \prod_{j=1}^{m} j^{j^{r-1}} \cdot r_r(n, m) \leq \binom{n-1+S_r(m)}{S_{r-1}(m)-1} \qquad \ldots \quad \ldots \quad (6)$$

where $S_k(m)$ denotes the sum of the kth powers of positive integers $\leq m$. Hence, if $m = o\left(n^{\frac{1}{2r+1}}\right)$,

$$\prod_{j=1}^{m} j^{j^{r-1}} \cdot r_r(n, m) \sim \binom{n-1}{S_{r-1}(m)-1} \qquad \ldots \quad \ldots \quad \ldots \quad (7)$$

Again, following Hardy (1940), it can be shown that for large n,

$$r_2(n) = \exp[\{C+o(1)\} n^{2/3}],$$

where $\qquad C^3 = \frac{27}{4}\{1^{-3}+2^{-3}+3^{-3}+4^{-3}+ \ldots\}$

so that C is nearly 2·01.
The tables that follow give the values of $r_2(n, m)$ for values of $m < n \leq 50$.

REFERENCES

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[illegible] G. H. (1940). "Ramanujan." 113-4.

TABLE OF VALUES OF $r_2(n, m)$.

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$m \downarrow$ / $n \leftarrow$	24	25	26	27	28
1	1	1	1	1	1
2	91	91	105	105	120
3	2147	2507	2937	3422	3947
4	16988	21454	27172	33938	42437
5	60713	81800	1 09468	1 45526	1 92288
6	1 32026	1 85413	2 59242	3 59395	4 95839
7	2 11756	3 06667	4 41249	6 30771	8 96344
8	2 83108	4 18175	6 14429	8 96439	13 01168
9	3 38584	5 07293	7 55612	11 18679	16 47023
10	3 78379	5 72173	8 60712	12 86679	19 13943
11	4 05274	6 16987	9 34126	14 06249	21 06168
12	4 22956	6 46585	9 83608	14 87585	22 39062
13	4 34123	6 65812	10 15835	15 41496	23 27878
14	4 41123	6 77838	10 36541	15 76286	23 86125
15	4 45353	6 85338	10 49426	15 98471	24 23400
16	4 47913	6 89850	10 57426	16 12215	24 47064
17	4 49375	6 92570	10 62220	16 20715	24 61667
18	4 50239	6 94118	10 65100	16 25791	24 70667
19	4 50695	6 95030	10 66734	16 28831	24 76025
20	4 50955	6 95510	10 67694	16 30551	24 79225
21	4 51081	6 95783	10 68198	16 31559	24 81031
22	4 51147	6 95915	10 68484	16 32087	24 82087
23	4 51170	6 95984	10 68622	16 32386	24 82639
24	4 51194	6 96008	10 68694	16 32530	24 82951
25		6 96033	10 68719	16 32605	24 83104
26			10 68745	16 32631	24 83179
27				16 32658	24 83206
28					24 83231

m / $n \rightarrow$	23	22	21	20	19
23	2 90783				
22	2 90760	1 86475			
21	2 90738	1 86453	1 18794		
20	2 90675	1 86432	1 18773	75278	
19	2 90555	1 86372	1 18753	75258	47330
18	2 90308	1 86258	1 18696	75239	47311
17	2 89876	1 86024	1 18588	75185	47293
16	2 89060	1 85616	1 18367	75083	47242
15	2 87684	1 84818	1 17983	74875	47146
14	2 85284	1 83558	1 17263	74515	46951
13	2 81336	1 81318	1 16059	73843	46615
12	2 74836	1 77652	1 13979	72725	45991
11	2 64528	1 71652	1 10595	70805	44959
10	2 48446	1 62258	1 05095	67703	43199
9	2 24446	1 47833	96660	62748	40379
8	1 90138	1 26998	84042	55314	36005
7	1 45154	98806	66714	44702	29685
6	93074	65311	45329	31304	21313
5	44726	32708	23711	17057	12146
4	13226	10333	7911	6097	4586
3	1812	1522	1282	1062	872
2	78	78	66	66	55
1	1	1	1	1	1

m ↓ \ ← n	29	30	31	32	33
1	1	1	1	1	1
2	120	136	136	153	153
3	4557	5243	5978	6825	7763
4	52423	64833	79354	97130	1 17824
5	2 52521	3 29792	4 28316	5 53478	7 11596
6	6 79175	9 26064	12 54360	16 91753	22 68267
7	12 66116	17 78692	24 [illegible]	34 54206	47 77165
8	18 75826	26 93988	38 [illegible]	54 62744	77 20947
9	24 11642	35 13096	50 91511	73 [illegible]	105 43419
10	28 36942	41 67746	61 [illegible]	89 [illegible]	129 19384
11	31 38150	46 53495	68 [illegible]	100 [illegible]	147 60476
12	33 [illegible]	49 97536	74 13982	109 [illegible]	161 99612
13	34 98046	52 32757	77 [illegible]	115 [illegible]	170 62863
14	35 91051	53 96298	80 [illegible]	119 [illegible]	177 17993
15	36 [illegible]	54 93288	82 [illegible]	122 39951	181 60133
16	36 96346	55 39900	83 [illegible]	124 20583	184 53085
17	37 21459	56 02205	83 [illegible]	125 47526	186 45134
18	37 36921	56 28827	84 [illegible]	126 12532	187 68956
19	37 46421	56 45148	84 [illegible]	126 59747	188 48129
20	37 52061	56 55148	84 [illegible]	126 89327	188 97829
21	37 55421	56 61070	84 97870	127 07366	189 28888
22	37 57313	56 64590	85 [illegible]	127 18366	189 47786
23	37 58447	56 66568	85 [illegible]	127 [illegible]	189 59286
24	37 58993	56 67720	85 [illegible]	127 28692	189 66654
25	37 59318	56 68329	85 11648	127 30842	189 70654
26	37 59474	56 68658	85 11642	127 32090	189 72290
27	37 59555	56 68820	85 11993	127 32738	189 73586
28	37 59583	56 68904	85 12161	127 33102	189 74258
29	37 59612	56 68933	85 12218	127 33276	189 74635
30		56 68963	85 12278	127 33366	189 74815
31			85 12309	127 33397	189 74908
32				127 33429	189 74940
33					189 74973

↑ m \ n →	18	17	16	15	14
18	29601				
17	29583	18334			
16	29566	18317	11297		
15	29518	18301	11281	6879	
14	29428	18256	11266	6864	4167
13	29246	18172	11224	6850	4153
12	28934	18003	11146	6811	4140
11	28358	17715	10990	6739	4104
10	27412	17187	10726	6596	4038
9	25812	16327	10246	6356	3908
8	23310	14887	9472	5924	3692
7	19546	12731	8220	5236	3308
6	14478	9658	6428	4193	2727
5	8578	5986	4133	2819	1902
4	3473	2551	1903	1364	992
3	722	582	466	376	294
2	55	45	45	36	36
1	1	1	1	1	1

m ↓ \ ← n	34	35	36	37	38
1	1	1	1	1	1
2	171	171	190	190	210
3	8771	9912	11172	12516	14028
4	1 42930	1 72048	2 06925	2 47179	2 95105
5	9 10563	11 59790	14 70798	18 57286	23 35838
6	36 28345	40 21954	53 20139	70 03151	91 83768
7	65 75350	90 08159	122 86196	166 84252	225 02210
8	108 64828	152 16527	212 22670	294 70004	407 61502
9	150 68833	214 42703	303 85441	428 80601	602 77426
10	186 72473	268 70493	385 15321	549 84296	782 02516
11	215 [illegible]	311 93702	450 77448	648 91182	930 70766
12	[illegible]	344 35040	500 55526	721 57270	1040 01481
13	250 96872	367 70178	536 73894	780 [illegible]	1131 11075
14	261 45711	384 04034	562 29146	820 42181	1192 71897
15	268 41539	395 24639	579 93851	847 [illegible]	1235 [illegible]
16	273 14266	402 79791	591 93619	866 86863	1265 34453
17	276 25777	407 82566	599 97192	879 61192	1285 18681
18	278 29123	411 12578	605 29857	888 15637	1299 02524
19	279 59824	413 27221	608 78203	893 78056	1308 01625
20	280 43164	414 61861	611 04143	897 44736	1313 [illegible]
21	280 95349	415 52308	612 48602	899 81973	1317 78645
22	281 27887	416 06978	613 40276	901 33341	1320 27191
23	281 47644	416 40995	613 97431	902 29152	1321 [illegible]
24	281 59644	416 61641	614 32927	902 88792	1322 85416
25	281 66694	416 74411	614 54102	903 25767	1323 47541
26	281 70851	416 81445	614 67402	903 48104	1323 85096
27	281 73476	416 85703	614 75016	903 61601	1324 09188
28	281 74520	416 88171	614 79496	903 69497	1324 23488
29	281 75216	416 89563	614 81990	903 74437	1324 31566
30	281 75606	416 90283	614 83430	903 76717	1324 36166
31	281 75792	416 90686	614 84174	903 78204	1324 38842
32	281 75888	416 90878	614 84590	903 78973	1324 40368
33	281 75921	416 90977	614 84788	903 79402	1324 41166
34	281 75955	416 91011	614 84890	903 79696	1324 41532
35		416 91046	614 84925	903 79711	1324 41812
36			614 84961	903 79747	1324 41929
37				903 79784	1324 41957
38					1324 41995

↑ m \ n →	13	12	11	10	9
13	2485				
12	2472	1479			
11	2460	1467	859		
10	2427	1456	848	5[illegible]	
9	2367	1426	838	490	282
8	2250	1372	811	484	27[illegible]
7	2058	1268	763	457	26[illegible]
6	1722	1100	672	415	24[illegible]
5	1263	827	528	337	20[illegible]
4	688	492	328	227	14[illegible]
3	226	178	132	97	7[illegible]
2	28	28	21	21	1[illegible]
1	1	1	1	1	1

2597 2606 2606 2603 2598 2597

2606 2605 2604 2603

39	40	41	42	n→ / m↓
1	1	1	1	1
210	231	231	253	2
15680	17444	19404	21540	3
3 50151	4 15124	4 89414	5 76540	4
29 26061	36 51688	45 40502	56 25896	5
119 89881	155 96450	202 05219	260 85672	6
303 86994	407 65475	544 80026	725 40570	7
561 49926	770 59691	1053 [illegible]	1435 27040	8
844 07058	1177 58492	1636 [illegible]	2267 56643	9
1108 06363	1564 49107	2201 14771	3086 44183	10
1330 67806	1894 23257	2688 [illegible]	3803 41864	11
1504 10620	2155 59225	3079 62177	4384 13290	12
1634 22951	2352 75025	3376 31649	4830 14042	13
1728 31494	2496 52040	3594 73018	5160 40959	14
1791 85179	2598 79580	3751 [illegible]	[illegible] 62729	15
1840 99819	2670 26988	3861 [illegible]	5567 21545	16
1872 58963	2719 48352	3937 [illegible]	5684 73475	17
1893 94573	2752 09667	3989 [illegible]	5765 70433	18
1908 24304	2775 54801	4023 17566	5820 88755	19
1917 70904	2790 60171	4048 92836	5858 46825	20
1923 92525	2800 54101	4064 [illegible]	5883 11289	21
1927 95873	2807 05323	4075 [illegible]	5899 67405	22
1930 55704	2811 27005	4081 [illegible]	[illegible]	23
1932 20800	2813 98133	4086 [illegible]	5917 06449	24
1933 24975	2815 70108	4089 [illegible]	5922 24769	25
1933 89585	2816 78450	4090 [illegible]	5925 18491	26
1934 29518	2817 45545	4092 [illegible]	5927 04224	27
1934 53570	2817 86957	4092 [illegible]	5928 20900	28
1934 68070	2818 11868	4093 [illegible]	5928 92965	29
1934 76530	2818 26868	4093 [illegible]	5929 37335	30
1934 81490	2818 35610	4093 [illegible]	5929 63964	31
1934 84242	2818 40730	4093 72326	5929 79964	32
1934 85826	2818 43568	4093 [illegible]	5929 89270	33
1934 86642	2818 45200	4093 80530	5929 94710	34
1934 87097	2818 46040	4093 82210	5929 97720	35
1934 87313	2818 46508	4093 83074	5929 99448	36
1934 87424	2818 46730	4093 [illegible]	5930 00336	37
1934 87462	2818 46844	4093 83783	5930 00830	38
1934 87504	2818 46883	4093 83900	5930 01064	39
	2818 46923	4093 83940	5930 01184	40
		4093 83981	5930 01225	41
			5930 01267	42

m ↑ / n →	8	7	6	5
8	160			
7	152	86		
6	145	79	18	
5	127	[illegible]	42	24
4	97	58	37	19
3	51	34	25	15
2	15	10	10	6
1	1	1	1	1

2602 2601 2600 2599

1	1	1	1	1
253	276	276	300	2
23808	26316	29028	31908	3
6 77794	7 91345	9 22483	10 74325	4
69 46852	85 49811	104 89000	128 28386	5
335 49494	431 07621	549 37160	699 57179	6
962 41635	1272 43087	1676 61254	2241 92978	7
1948 33405	2635 [illegible]	3554 20354	4777 09946	8
3130 40182	4307 27018	5907 46711	8076 72749	9
4313 43932	6069 07423	8344 90336	11553 70959	10
5363 39768	7539 72904	10567 02392	14766 16521	11
6222 91466	8806 40546	12425 57162	17482 08509	12
6889 03209	9796 66226	13891 49166	19642 90570	13
7385 95663	10541 05024	15001 52898	21291 37868	14
7746 85738	11085 15829	15818 30378	22512 00658	15
8003 78098	11474 91573	16406 73234	23396 59978	16
8183 97486	11749 75528	16824 04749	24027 23637	17
8308 78452	11941 22848	17116 25841	24471 19143	18
8394 37553	12073 20210	17318 79081	24780 39572	19
8452 49913	12163 36840	17457 84721	24993 84332	20
8491 65216	12224 41708	17552 56162	25139 93045	21
8517 78684	12265 43927	17616 52651	25239 17586	22
8535 10078	12292 76189	17659 41576	25306 05342	23
8546 45998	12310 82861	17687 92632	25350 80742	24
8553 86023	12322 66111	17706 74582	25380 50592	25
8558 62707	12330 35737	17719 05162	25400 07820	26
8561 67726	12335 [illegible]	17727 04389	25412 85730	27
8563 60338	12338 47671	17732 17741	25421 [illegible]	28
8564 81181	12340 46562	17735 45354	25426 [illegible]	29
8565 55731	12341 71572	17737 51724	25429 [illegible]	30
8566 01580	12342 48607	17738 80901	25431 [illegible]	31
8566 29068	12342 95935	17739 60421	25433 31747	32
8566 45568	12343 24282	17740 09228	25434 13752	33
8566 55156	12343 41282	17740 38434	25434 64038	34
8566 60756	12343 51152	17740 55934	25434 94163	35
8566 63852	12343 56912	17740 66086	25435 12163	36
8566 65628	12343 60094	17740 72006	25435 22537	37
8566 66540	12343 61918	17740 75274	25435 28617	38
8566 67047	12343 62854	17740 77146	25435 31971	39
8566 67287	12343 63374	17740 78106	25435 33891	40
8566 67410	12343 63620	17740 78639	25435 34875	41
8566 67452	12343 63746	17740 78891	25435 35421	42
8566 67495	12343 63789	17740 79020	25435 35679	43
	12343 63833	17740 79064	25435 35811	44
		17740 79109	25435 35856	45
			25435 35902	46

m ↑ / n →	4	3	2	1
4	13			
3	9	6		
2	6	3	3	
1	1	1	1	1

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47	48	49	50	← n / m ↓
1	1	1	1	1
300	325	325	351	2
35064	38469	42069	45999	3
12 46034	14 43675	16 66409	19 21549	4
156 42512	190 18801	230 58592	278 80409	5
887 89182	1123 57406	1417 38607	1783 01142	6
2882 58242	3761 96535	4894 57175	6349 48264	7
6400 31762	8549 11171	11384 88189	15117 52576	8
11008 75106	14960 43757	20271 41589	27389 87134	9
15948 61891	21951 97322	30129 37489	41239 49246	10
20574 54714	28587 31039	39611 77639	54741 03007	11
24527 24998	34318 07140	47888 93614	66653 11966	12
27699 77492	38957 57302	54648 53[illegible]28	76465 26317	13
30137 65589	42548 93905	59918 69875	84170 88823	14
31954 44614	45241 99240	63895 48545	90021 07453	15
33278 43454	47215 73488	66826 00425	94355 88973	16
34227 42979	48637 61879	68948 02823	97510 19859	17
34898 76130	49648 54478	70463 80220	99774 07884	18
35368 72383	50359 50977	71534 91143	1 01380 88585	19
35694 67063	50855 05077	72284 78783	1 02510 96545	20
35918 94181	51197 57736	72805 62374	1 03299 26786	21
36072 02925	51432 61381	73164 62744	1 03845 21612	22
36175 79582	51592 68254	73410 39532	1 04220 63201	23
36245 58374	51700 96634	73577 43448	1 04477 11029	24
36292 20249	51773 66209	73690 23298	1 04651 11554	25
36323 08893	51822 14559	73765 83656	1 04768 42598	26
36343 41399	51854 21997	73816 18481	1 04846 93739	27
36356 66639	51875 29781	73849 44713	1 04899 15039	28
36365 25068	51889 02351	73871 27775	1 04933 69065	29
36370 75088	51897 96381	73885 47675	1 04956 18405	30
36374 25295	51903 58735	73894 65306	1 04970 85635	31
36376 45423	51907 20239	73900 51994	1 04980 32867	32
36377 82934	51909 47246	73904 24786	1 04986 37889	33
36378 67424	51910 88924	73906 58681	1 04990 21987	34
36379 19189	51911 75899	73908 04526	1 04992 62752	35
36379 50113	51912 29143	73908 93986	1 04994 12764	36
36379 68613	51912 60926	73909 48709	1 04995 04709	37
36379 79329	51912 79926	739[illegible]9 81351	1 04995 60911	38
36379 85569	51912 90924	73910 00851	1 04995 94412	39
36379 89009	51912 97324	73910 12131	1 04996 14412	40
36379 90977	51913 00850	73910 18691	1 04996 25974	41
36379 91985	51913 02866	73910 22303	1 04996 32694	42
36379 92544	51913 03898	73910 24367	1 04996 36392	43
36379 92808	51913 04470	73910 25423	1 04996 38504	44
36379 92943	51913 04740	73910 26008	1 04996 39584	45
36379 92989	51913 04878	73910 26284	1 04996 40182	46
36379 93036	51913 04925	73910 26425	1 04996 40464	47
* *	51913 04973	73910 26473	1 04996 40608	48
* *	* *	73910 26522	1 04996 40657	49
* *	* *	* *	1 04996 40707	50