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A FORMULA FOR $L(n)$.

BY

HANSRAJ GUPTA, *Hoshiarpur*

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1. In what follows, all small letters denote integers ≥ 0; the letter m is used to denote only quadratfrei numbers; p denotes any prime; p_s the sth prime with $p_1 = 2$. $Q(p_s)$ denotes the set of quadratfrei integers none of whose prime factors exceeds p_s. The set of odd quadratfrei numbers satisfying the same condition is denoted by $q(p_s)$. Unity shall be considered quadratfrei for the purposes of this paper. $[a/b]$ stands as usual for the greatest integer $\leq a/b$. $L(n)$ and $\lambda(n)$ are Liouville's functions defined by the relations:

$$\lambda(0) = 0,\ \lambda(1) = 1,\ \lambda(pn) = -\lambda(n)$$

and
$$L(n) = \lambda(1) + \lambda(2) + \lambda(3) + \ldots + \lambda(n).$$

We write $L_s(n)$ for $\sum_{\substack{t \leq n \\ (t,\, p_1 p_2 p_3 \ldots p_s) = 1}} \lambda(t)$. Thus $L(n)$ and $L_0(n)$ mean the same thing.

Also $\pi(a,\ b)$ = the number of primes $> a$ and $\leq b$, when $0 \leq a < b$;

$= 0$, when $a = b$ and also when $b = 0$;

$= -1$, when $0 < b < a$.

2. With these notations, we have

$$L(n) + \sum_{v=1}^{r} L_{v-1}\left[\frac{n}{p_v}\right] = 1 \tag{1}$$

where p_r is the greatest prime not exceeding n.

Let p_i be the greatest prime not exceeding $n^{\frac{1}{2}}$, p_r the greatest prime not exceeding n and p_k the greatest prime $\leq (n/2)^{\frac{1}{2}}$. Then, we have

$$L_{v-1}\left[\frac{n}{p_v}\right] = -\pi\left(p_v, \left[\frac{n}{p_v}\right]\right), \text{ for every } v > i.$$

Also, $$L(n) + \sum_{v=1}^{t} L_{v-1}\left[\frac{n}{p_v}\right] = \sum_{m = Q(p_t)} L\left[\frac{n}{m}\right], \ t \geq 1.$$

Hence $$\sum_{m = Q(p_t)} L\left[\frac{n}{m}\right] = 1 + \sum_{v=t+1}^{r} \pi\left(p_v, \left[\frac{n}{p_v}\right]\right), \ t \geq i. \tag{2}$$

This implies that

$$\sum_{m = q(p_t)} L\left[\frac{n}{2m}\right] = 1 + \sum_{v=t+1}^{r} \pi\left(p_v, \left[\frac{n}{2p_v}\right]\right), \ t \geq k. \tag{3}$$

Subtracting (3) from (2), we get

$$\sum_{m = q(p_t)} \left\{L\left[\frac{n}{m}\right] - L\left[\frac{n}{4m}\right]\right\} = \sum_{v=t+1}^{r} \left\{\pi\left(p_v, \left[\frac{n}{p_v}\right]\right) - \pi\left(p_v, \left[\frac{n}{2p_v}\right]\right)\right\}, \ t \geq i. \tag{4}$$

In particular when $t = i$, the right side of (3)

$$= \sum_{v=i+1}^{i+k} \pi\left(\left[\frac{n}{2p_v}\right], \left[\frac{n}{p_v}\right]\right) + \sum_{v=i+k+1}^{i+j} \pi\left(p_v, \left[\frac{n}{p_v}\right]\right) - \pi(p_{i+j}, n) + \pi\left(p_{i+k}, \left[\frac{n}{2}\right]\right). \tag{5}$$

Now

$$\pi(p_{i+j}, n) = \sum_{m = Q(p_t)} \lambda(m)\left[\frac{n}{m}\right] - \sum_{v=t+1}^{i+j-t} \pi\left(p_v, \left[\frac{n}{p_{v}}\right]\right) + \tfrac{1}{2}(i+j-t-1)(i+j-t-2) - 2,\ t \geq i. \tag{6}$$

This is a small variation of Meissel's formula for $\pi([\sqrt{n}], n)$. After easy simplifications, we now obtain

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$$+\sum_{m=q(p_i)}\left\{L\left[\frac{n}{m}\right]-L\left[\frac{n}{4m}\right]+\lambda(m)\left(\left[\frac{n}{4m}\right]+\delta_m\right)\right\}$$

$$-\left\{\sum_{r=1}^{k}\pi\left(\left[\frac{n}{2p_{i+r}}\right],\left[\frac{n}{p_{i+r}}\right]\right)+\sum_{r=k+1}^{j}\pi\left(p_{i+r-1},\left[\frac{n}{p_{i+r}}\right]\right)\right\}, \qquad (7)$$

where $\delta_m = 1$ or 0 according as $[n/m]$ is odd or even.

From a table of values of $L(n)$ for values of n up to n_1 say, it would be possible to calculate directly with the help of (7), the value of $L(n)$ for any n up to $3n_1$. The writer calculated a few years back, a table of values of $L(n)$ for values of n up to 20,000. With the help of this and the set of numbers $q(31)$, I have now been able to verify within a few hours that $L(48512) = -2$. All that was needed was a table of primes up to 1311.

3. It may be interesting to note that for the evaluation of the left side of (7), all the members of $q(p_i)$ will not be needed; for denoting

$$L(r)-L[r/4]-([r/4]+\delta_r) \text{ by } E_r$$

and

$$L(r)-L[r/4]+([r/4]+\delta_r) \text{ by } F_r,$$

with $\delta_r = 1$ or 0, according as r is odd or even, we find that $E_r = 0$ when $r = 1$ or 2; and $F_r = 0$ when $2 \leqslant r \leqslant 8$. Moreover $E_3 = E_4 = -2$, and $F_1 = F_9 = F_{10} = F_{11} = \ldots = F_{14} = 2$.

These and similar considerations introduce several simplifications in the process of computation. It would be best to have the table $q(p_i)$ in two parts—the first consisting of those m's for which $\lambda(m) = -1$, and the second of those for which $\lambda(m) = 1$, the numbers in each part being serially arranged.

For ready reference a table of values of m for values of n up to 50672 is given below:

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Part I. $\lambda(m) = -1$.

3	231	561	957	1309	1771	2635	3565	5123	8671
5	255	595	969	1311	1885	2639	3689	5681	9269
7	273	609	1001	1463	1955	2697	3857	5797	9367
11	285	627	1015	1479	2001	2717	4123	6061	9889
13	345	651	1023	1495	2015	2737	4147	6293	10013
17	357	663	1045	1547	2093	2755	4199	6409	11339
19	385	665	1085	1581	2139	2821	4301	6479	11687
23	399	715	1105	1595	2185	2945	4433	6851	12121
29	429	741	1131	1615	2233	3059	4495	7163	12673
31	435	759	1173	1653	2261	3289	4669	7337	13547
105	455	805	1209	1705	2387	3335	4807	7429	15015
165	465	897	1235	1729	2431	3451	4991	7657	15283
195	483	935	1265	1767	2465	3553	5083	7813	

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Part II. $\lambda(m) = 1$.

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1	203	2145	6555	11165	17391	25415	37145
15	209	2415	6699	11305	17765	26013	37961
21	217	2805	6783	11571	18183	27115	38019
33	221	3003	7161	11935	18445	27807	38285
35	247	3045	7293	12155	18879	28101	39215
39	253	3135	7315	12369	19019	28405	39767
51	299	3255	7395	12441	19227	28985	40579
55	319	3315	7735	12597	19285	29029	40641
57	323	3705	7905	12903	19437	29393	42427
65	341	3795	7917	13195	20553	29667	43355
69	377	3927	8151	13299	20615	30039	44803
77	391	4389	8211	13485	20735	30107	45353
85	403	4485	8265	13585	20995	30305	45849
87	437	4641	8463	13685	21189	31031	46189
91	493	4785	8645	14007	21505	31465	46345
93	527	4845	8835	14105	22011	32015	46835
95	551	5005	8855	14421	22165	32305	47957
115	589	5115	9177	14973	22287	33649	49445
119	667	5187	9867	15249	22671	34017	50065
133	713	5313	10005	15295	23023	34255	50141
143	899	5655	10353	16269	23345	35061	
145	1155	5865	10465	16445	23529	35581	
155	1365	6045	10659	17017	24035	35815	
161	1785	6279	10695	17043	24871	36363	
187	1995	6545	11067	17255	24955	36685	

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