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COMPLETE CONNECTIVES FOR THE 3-VALUED PROPOSITIONAL CALCULUS

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1. Introduction

The problem of finding complete connectives for the m-valued propositional calculus may be given a purely algebraic formulation as the problem of determining complete generators of the composition algebra on m marks.

The elements of such an algebra are functions $f(x_1, x_2, \ldots, x_n)$, whose n variables $x_1, \ldots, x_n$ range over a fixed finite set M consisting of m marks and whose values belong to the same set; that is, functions which map $M \times M \times \ldots \times M$ into M. Note that, throughout this paper, m will be used exclusively for the number of marks and n for the number of arguments in a function.

The fundamental algebraic operation on the elements is composition. Let $\xi_1, \xi_2, \ldots, \xi_\nu$ be variables which range over the same set of marks; ν may be greater than, equal to, or less than n. Suppose that $f_i(\xi_1, \xi_2, \ldots, \xi_\nu)$ $(i = 1, 2, \ldots, n)$ are n given functions and that each x_i is restricted by being $f_i(\xi_1, \xi_2, \ldots, \xi_\nu)$. (Of course any—or all—of the ξ_j may be absent from any f_i.) Then $f(x_1, x_2, \ldots, x_n)$ becomes a function $f'(\xi_1, \xi_2, \ldots, \xi_\nu)$, say, of the variables $\xi_1, \ldots, \xi_\nu$, and we may write

$$f'(\xi_1, \xi_2, \ldots, \xi_\nu) = f[f_1, f_2, \ldots, f_n](\xi_1, \xi_2, \ldots, \xi_\nu),$$

or, omitting the argument set if clear from the context,

$$f' = f[f_1, f_2, \ldots, f_n].$$

f' is then said to have been *generated* by the $n+1$ elements on the right. For purposes of distinction, round brackets can conveniently be used for an argument set and square brackets for functional composition.

A complete generator of this algebra is a function which by itself will generate by repeated composition every function of any number of arguments which belongs to the algebra, starting with the singulary function θ, where $\theta(x) = x$. The basic problem of the theory is the discovery of necessary and sufficient conditions which a complete generator must satisfy.

Post's theorem ((5) 107) solves this problem for the case $m = 2$. (A simpler proof of Post's theorem has been supplied by R. A. Cuningham-Green (1); a description of this proof can be found in an article by Professor R. L. Goodstein (3).)

The theorem proved in the present paper establishes conditions which are necessary and sufficient for a function $f(x_1, x_2, \ldots, x_n)$ to be a complete generator of the composition algebra on 3 marks (i.e. a complete connective for the 3-valued propositional calculus), and so gives a complete solution to the above problem for the next value, $m = 3$.

Necessary and sufficient conditions for the particular case $m = 3$, $n = 2$ were given by Martin (4), who used them to enumerate the complete binary connectives of the 3-valued propositional calculus. The set of conditions below, however, is somewhat simpler than Martin's, as well as being discriminative for all values of n. (For the case $n = 2$, Martin's original conditions have been simplified by Foxley (2), who showed that one of Martin's conditions was redundant.)

Considerable use will be made of the important theorem of Arto Salomaa ((6) 21) that an n-place function ($n \geqslant 2$) will be a complete generator if and only if it generates all 1-place functions. (It is well known that no 1-place function can be a complete generator.) In other respects, however, the proof in this paper is self-contained. It must be pointed out that several writers (notably Salomaa ((6) (7))) have obtained partial results which are implicit in the proof below, but it was felt that if these had been used as starting points the proof would have lost something in directness and cohesion.

2. Notation ($m = 3$)

1. The 3 marks may be denoted by 0, 1, 2, but since all the following arguments depend on permutational rather than numerical properties the choice of a, b, c for the marks has been preferred, and these letters are used throughout *exclusively* with this meaning.

2. If g denotes any one of the marks a, b, c, then $\bar{g}$ denotes either of the 2 marks which is not g; thus, $x = \bar{a}$ means that $x = b$ or $x = c$, $(x_1, x_2) = (\bar{a}, \bar{a})$ means that (x_1, x_2) has one of the 4 values (b, b), (b, c), (c, b), (c, c), and so on.

3. If g is one of the marks, then g' denotes the image of g under the particular cyclic permutation $a \to b \to c \to a$, and g'' denotes the image of g'.

4. In order to make a later notation easier to print, the argument set of the function will in future always be written vertically, thus

$$f\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

(and not horizontally, as was done in the introduction above). The context will, in fact, always be sufficiently clear for this to be abbreviated to

$$f(x_i),$$

and the suffix i will always be understood to range from 1 to n. The argument set itself will be written $\{x_i\}$. Also

$$f\begin{pmatrix} g \\ g \\ \vdots \\ g \end{pmatrix}$$

will be abbreviated to

$$f(g).$$

Similarly, $f(\bar{a})$ refers to the set of 2^n values represented by

$$f\begin{pmatrix} \bar{a} \\ \bar{a} \\ \vdots \\ \bar{a} \end{pmatrix};$$

it often happens that such a set of values can be dealt with collectively.

5. Corresponding capital letters will very often be used to represent the value of the function for a given argument set, thus

$$P = f(p_i), \quad f(e_i) = E,$$

and so on. In particular, A, B, C, will be used (exclusively) to denote the values of the function on the repeated marks, that is

$$A = f(a), \quad \text{i.e.} \quad f\begin{pmatrix} a \\ a \\ \vdots \\ a \end{pmatrix}, \quad \text{etc.}$$

These values A, B, C play a prominent part in the classification of f, and a function taking these values on the repeated marks will be referred to as a function of *type* $[A\ B\ C]$. Thus, if $f(a) = b$, $f(b) = a$, $f(c) = a$, f will be said to be of type $[b\ a\ a]$.

6. A similar square-bracket notation will be used to exhibit the value set of any singulary function. The singulary functions themselves will be denoted by φ, χ, ψ, and θ. For the present purpose, there is no need to maintain any close distinction between a singulary function and its value set, and we shall write

$$\varphi = [p\,q\,r]$$

to denote that the value set of φ is $[p\,q\,r]$, i.e. that $\varphi(a) = p$, $\varphi(b) = q$, $\varphi(c) = r$.

In particular, θ will always be reserved (as above) for the singulary function $\theta(x) = x$, i.e.

$$\theta = [a\,b\,c].$$

7. When discussing the generation of singulary functions, it will be convenient to telescope 3 statements such as

$$f(p_i) = P, \quad f(q_i) = Q, \quad f(r_i) = R$$

into the composite statement

$$f[p_i\,q_i\,r_i] = [P\,Q\,R].$$

(This was the reason for selecting notation (4) originally.)

Thus $[p_i\,q_i\,r_i]$ is to be thought of as a shortened form of the $n \times 3$ array

$$\begin{matrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ \vdots & \vdots & \vdots \\ p_n & q_n & r_n \end{matrix}$$

and this must be kept in mind when reference is made to rows and columns of $[p_i\,q_i\,r_i]$.

If φ_i denotes the singulary function $[p_i\,q_i\,r_i]$, $f[p_i\,q_i\,r_i]$ may be written alternatively as

$$f\begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_n \end{bmatrix}, \quad \text{or} \quad f[\varphi_i],$$

abbreviating as in (4).

The rows of the above $n \times 3$ array are the value sets of n singulary functions, and the columns are 3 argument sets of f. We shall be concerned presently with the problem of generating all the singulary functions of 3 marks, starting with θ. We obviously do not want to have to specify the values of f for *all* the 3^n possible argument sets, but rather to show that f is a complete generator by fixing its values for as *few* argument sets as possible. At each stage, we shall be interested in generating new singulary functions $[P\ Q\ R]$, using only singulary functions $[p_i\ q_i\ r_i]$, all of which have already been constructed. At each step at which f is used, therefore, it is necessary to ensure that the columns of the above array are confined to the argument sets for which the value of f has actually been specified, while the rows of the array include only the value sets of singulary functions already generated.

8. *Iteration* of a singulary function φ by f means the derivation of the new singulary function

$$\varphi^{(1)} = f[\varphi], \quad \text{i.e.} \quad f\begin{bmatrix} \varphi \\ \varphi \\ \vdots \\ \varphi \end{bmatrix}.$$

The iterate of any function φ by f will always be denoted by $\varphi^{(1)}$, the iterate of $\varphi^{(1)}$ by $\varphi^{(2)}$, and so on.

Note that, since $[A\ B\ C]$ is $f[a\ b\ c]$,

$$[A\ B\ C] = \theta^{(1)}.$$

3. Definitions (general value of m)

1. *Conjugation*

Under any given permutation of the marks, f will be transformed into a function f_1 which may or may not be different from f. Let $f_1, f_2, \ldots, f_l$ be the complete set of distinct functions obtainable from f by applying in turn all $m!$ permutations to the set of marks. These functions are called the *conjugates of* f. Their total number l will always divide $m!$.

If, in particular, $l = m!$, i.e. if there is no permutation of the marks other than the identity which leaves f unaltered, f will be called *fully conjugated.*

EXAMPLES ($m = 3$, $n = 2$).

E1

	$x_2 \backslash x_1$	a	b	c																
	a	b	b	c	b	c	b	c	c	a	c	b	c	b	a	b	c	c	b
($l = 6$)	b	c	a	a	a	a	c	c	c	a	b	a	a	a	c	c	c	c	b
	c	c	a	a	b	c	b	a	b	b	b	a	a	b	a	b	a	b	a

E2

	$x_2 \backslash x_1$	a	b	c			
	a	b	b	c	c	b	c
($l = 2$)	b	a	c	c	a	a	c
	c	a	b	a	a	b	b

In the particular case $m = 2$, two distinct conjugates are described as duals, and a function *not* fully conjugated is called self-dual.

2. *Closure*

A subset S of the set M of marks will be said to be *closed under f* if

$$f(x_i) \in S \quad \text{whenever} \quad x_i \in S \quad \text{for} \quad i = 1, 2, \ldots, n.$$

EXAMPLE. If f is of type $[b\ a\ a]$ and if

$$f(x_i) = \bar{c} \quad \text{whenever} \quad x_i = \bar{c},$$

the set $\{a, b\}$ is closed under f.

3. *Invariance*

If $\mathscr{P}$ is any partition of the marks into disjoint classes, we shall, following Martin (4), write

$$x \sim y\ (\mathscr{P})$$

to denote that x and y belong to the same class in $\mathscr{P}$.

A partition $\mathscr{P}$ of the set of marks will be said to be *invariant under f* if

$$f(x_i) \sim f(y_i)\ (\mathscr{P}) \quad \text{whenever} \quad x_i \sim y_i\ (\mathscr{P}) \quad \text{for} \quad i = 1, 2, \ldots, n.$$

EXAMPLE. The partition $a|bc$ is invariant under the function in Example above, since

$$f\binom{a}{b} \sim f\binom{a}{c},$$

$$f\binom{b}{a} \sim f\binom{c}{a},$$

$$f\binom{b}{b} \sim f\binom{b}{c} \sim f\binom{c}{b} \sim f\binom{c}{c}.$$

4. *δ-Function*

A *subset* of the marks will be called *proper* if it is neither the empty set nor the complete set of marks.

A *partition* of the marks will be called *improper* either if each mark belongs to a different class in the partition or if every mark belongs to the same class, and *proper* otherwise.

A function f will be called a *δ-function* if

(1) there is no proper subset of the marks which is closed under f, and

(2) there is no proper partition of the marks which is invariant under f.

(This extension of the concept of a δ-function will be seen to harmonise with the usual definition ((5) 43) of a δ-function for the particular case $m = 2$, since the second condition above is nugatory when $m = 2$.)

These two δ-function properties and the property of being fully conjugated may be further illustrated and shown to be independent by the following simple examples, again taking $m = 3$, $n = 2$, and this time using commutative functions.

f	$f\binom{a}{a}$	$f\binom{b}{b}$	$f\binom{c}{c}$	$f\binom{b}{c} = f\binom{c}{b}$	$f\binom{c}{a} = f\binom{a}{c}$	$f\binom{a}{b} = f\binom{b}{a}$
$\mathscr{F}_1$	b	a	a	b	c	a
$\mathscr{F}_2$	b	a	a	a	b	c
$\mathscr{F}_3$	b	c	a	a	b	c

$\mathscr{F}_1$ and $\mathscr{F}_2$ are fully conjugated but are not δ-functions, because under $\mathscr{F}_1$ the subset $\{a, b\}$ is closed (though no partition is invariant under $\mathscr{F}_1$), while under $\mathscr{F}_2$ the partition $a \mid bc$ is invariant (though no subset is closed). $\mathscr{F}_3$ is a δ-function but is not fully conjugated, as it is unaltered by the cyclic permutations $g \to g'$ and $g \to g''$.

4. Necessary conditions (general value of m)

It is clear that, even for the general value of m, *any complete generator must be fully conjugated and must be a δ-function.* (This observation is really only a synthesis, using a new terminology, of various selected necessary conditions which have already been noticed by other writers.)

For, if a certain function f' is generated from θ by a particular composition sequence, the application of a given permutation to the marks throughout the process must yield a function conjugate to f'. Hence, if a given function is not fully conjugated it cannot possibly generate by itself those functions which do have $m!$ conjugates, and hence it cannot be a complete generator.

Similarly, if a given function leaves a subset S closed or a partition $\mathscr{P}$ invariant, every repetition of the function preserves that property. Such a function, therefore, can never generate by itself those functions which do *not* possess that property, and when S or $\mathscr{P}$ is proper this means that not all functions can be constructed.

When $m = 2$, the above conditions are equivalent to Post's conditions that f is a non-self-dual δ-function, and we shall now turn to the case $m = 3$ for further investigation.

5. The algebra of 3 marks

The above necessary conditions impose certain immediate limitations on the properties of a complete generator. First, $f(g) \neq g$ for any g, from the closure requirement, and so $A \neq a$, $B \neq b$, $C \neq c$. Hence there are, essentially, only 2 distinct types of function, type $[b\ a\ a]$ and type $[b\ c\ a]$.

1. *Type* $[b\ a\ a]$

This has been chosen as the representative of the 6 possible cases

$$[A\ B\ C] = [b\ a\ a],\ [c\ a\ a],\ [b\ c\ b],\ [b\ a\ b],\ [c\ c\ a],\ [c\ c\ b],$$

which are all basically the same.

A function of type $[b\ a\ a]$ is always fully conjugated, but such a function may fail to be a δ-function, as in examples $\mathscr{F}_1$ and $\mathscr{F}_2$ above. This will happen if and only if either

(1) the subset $\{a, b\}$ is closed under f, i.e.

$$f(x_i) = \bar{c} \quad \text{whenever} \quad x_i = \bar{c}, \quad \text{or}$$

(2) the partition $a\,|\,bc$ is invariant under f, i.e.

$$f(x_i) \sim f(y_i) \quad \text{whenever} \quad (x_i, y_i) = (a, a) \quad \text{or} \quad (\bar{a}, \bar{a}).$$

(In particular, this means that $f(x_i) = a$ whenever $x_i = \bar{a}$, since $f(b) = f(c) = a$.)

Clearly, with a function of type $[b\ a\ a]$, no subsets of the marks other than $\{a, b\}$ and no partitions other than $a\,|\,bc$ need be considered. (At first sight, it might be thought that the partition $ab\,|\,c$ could also be invariant under f. It will be seen, however, that in that case it would happen that $f(x_i) = \bar{c}$ whenever $x_i = \bar{c}$, so that the subset $\{a, b\}$ would be closed under f and the function would already have been rejected for that reason.)

2. *Type* $[b\ c\ a]$

The 2 cases $[A\ B\ C] = [b\ c\ a]$ and $[c\ a\ b]$ are effectively equivalent, and the first has been selected as typical.

A function of type $[b\ c\ a]$ is always a δ-function, since, whatever values the function $f(x_i)$ takes when the x_i are not all the same, no subset of the marks can be closed under f and no partition of the marks can be invariant under f.

A function of type $[b\ c\ a]$ may, however, not be fully conjugated, as in example $\mathscr{F}_3$ above. When this happens, it must be the cyclic permutations of the marks that leave the function unaltered.

THEOREM. *Necessary and sufficient conditions for a given n-place function to be a complete generator of the composition algebra on 3 marks are that $n \geqslant 2$ and that the function is a fully conjugated δ-function.*

The necessity of these conditions (for the general value of m) has already been explained. The sufficiency of the conditions for the case $m = 3$ will now be proved by showing that any function which satisfies them will generate all the singulary functions of 3 marks, from which the result will follow from Salomaa's theorem, mentioned earlier. The proof is in two parts.

Proof—Part 1. *A function of type* $[b\ a\ a]$

Such a function is always fully conjugated; by hypothesis, it must be a δ-function.

The method of proof adopted for this type of function concentrates first on establishing that the functions which are the other permutations of $[b\ a\ a]$, namely $[a\ b\ a]$ and $[a\ a\ b]$, can also be generated in each of various cases that arise. By this means, the last stage of the proof can be completed very quickly, taking all the cases together.

Now

$$\begin{aligned} \theta &= [a\ b\ c], \\ \theta^{(1)} &= [b\ a\ a], \\ \theta^{(2)} &= [a\ b\ b], \end{aligned}$$

and to start with these are the only singulary functions available.

Since the partition $a \mid bc$ is, by hypothesis, not invariant under f, there must exist (at least) 2 sets $\{r_i\}$, $\{s_i\}$ with the property that

$$r_i \sim s_i \quad \text{and} \quad f(r_i) \nsim f(s_i),$$

i.e. for all i,

$$\text{either} \quad (r_i, s_i) = (a, a) \quad \text{or} \quad (r_i, s_i) = (\bar{a}, \bar{a})$$

while, say,

$$f(r_i) = a \quad \text{and} \quad f(s_i) = \bar{a}.$$

If $(r_i, s_i) = (a, a)$, define $p_i = b$, $q_i = a$, and if $(r_i, s_i) = (\bar{a}, \bar{a})$, define $p_i = a$, $q_i = b$. Then, since $[b\ a\ a]$, $[a\ b\ b]$, $[a\ b\ c]$ are available, we can generate $[p_i\ q_i\ r_i]$ and $[p_i\ q_i\ s_i]$ in all cases. Let

$$f(p_i) = P \quad \text{and} \quad f(q_i) = Q.$$

Then

$$f[p_i\ q_i\ r_i] = [P\ Q\ a] = \varphi_1 \quad \text{(say)}$$

and

$$f[p_i\ q_i\ s_i] = [P\ Q\ \bar{a}] = \varphi_2 \quad \text{(say)}.$$

(1) $(P, Q) = (a, a)$ or $(\bar{a}, \bar{a})$.

If $(P, Q) = (a, a)$,

$$\varphi_2 = [a\ a\ \bar{a}], \quad \varphi_2{}^{(1)} = [b\ b\ a], \quad \varphi_2{}^{(2)} = [a\ a\ b],$$

(since $f(b)$ and $f(c)$ are both equal to a). But $[p_i\ a\ q_i]$ is $[a\ a\ b]$ or $[b\ a\ a]$, so we can now generate

$$f[p_i\ a\ q_i] = [P\ b\ Q] = [a\ b\ a],$$

giving the third permutation of $[b\ a\ a]$. (See the observation at the beginning of this part of the proof.)

Similarly, if $(P,Q) = (\bar{a},\bar{a})$,

$$\varphi_1 = [\bar{a}\;\bar{a}\;a], \quad \varphi_1{}^{(1)} = [a\;a\;b], \quad \varphi_1{}^{(2)} = [b\;b\;a].$$

But $[p_i\;b\;q_i]$ is $[b\;b\;a]$ or $[a\;b\;b]$, enabling us to construct

$$f[p_i\;b\;q_i] = [P\;a\;Q] = [\bar{a}\;a\;\bar{a}],$$

the iterate of which is $[a\;b\;a]$, completing the trio.

(2) $(P,Q) = (a,\bar{a})$ or $(\bar{a},a)$.

If $(P,Q) = (a,\bar{a})$,

$$\varphi_1 = [a\;\bar{a}\;a], \quad \varphi_1{}^{(1)} = [b\;a\;b], \quad \varphi_1{}^{(2)} = [a\;b\;a],$$

while if $(P,Q) = (\bar{a},a)$,

$$\varphi_2 = [\bar{a}\;a\;\bar{a}], \quad \varphi_2{}^{(1)} = [a\;b\;a], \quad \varphi_2{}^{(2)} = [b\;a\;b].$$

To complete (2), the second part of the δ-function hypothesis is needed. Since the subset $\{a,b\}$ is not closed under f, there must be at least one argument set $\{d_i\}$, with each d_i equal to a or b, such that

$$f(d_i) = c.$$

Define also the set $\{e_i\}$ such that $e_i = a$ or b according as $d_i = b$ or a respectively, and let

$$f(e_i) = E.$$

Now

$$\begin{aligned} [e_i\;d_i\;d_i] &\quad \text{is} \quad [a\;b\;b] \quad \text{or} \quad [b\;a\;a],\\ [e_i\;d_i\;a] &\quad \text{is} \quad [a\;b\;a] \quad \text{or} \quad [b\;a\;a],\\ [e_i\;d_i\;b] &\quad \text{is} \quad [a\;b\;b] \quad \text{or} \quad [b\;a\;b], \end{aligned}$$

and so we can construct

$$\begin{aligned} f[e_i\;d_i\;d_i] &= [E\;c\;c],\\ f[e_i\;d_i\;a] &= [E\;c\;b],\\ f[e_i\;d_i\;b] &= [E\;c\;a]. \end{aligned}$$

Suppose first that $E = a$. Then we have the 5 functions ($[b\;a\;a]$, $[a\;b\;b]$, $[a\;b\;c]$, $[a\;c\;b]$, $[a\;c\;c]$) necessary to generate

$$\begin{aligned} f[p_i\;r_i\;s_i] &= [P\;a\;\bar{a}],\\ f[p_i\;s_i\;r_i] &= [P\;\bar{a}\;a]. \end{aligned}$$

Thus, whether $P = a$ or $P = \bar{a}$, we can obtain (by iteration, if necessary) the function $[a\;a\;b]$.

Suppose, on the other hand, that $E = \bar{a}$. Then $f[E\;c\;s_i] = [a\;a\;\bar{a}]$, and again $[a\;a\;b]$ can be produced.

This completes the first stage of this part of the proof, giving, in all cases, the functions

$$[b\;a\;a], \quad [a\;b\;a], \quad [a\;a\;b]$$

and their iterates

$$[a\ b\ b], \quad [b\ a\ b], \quad [b\ b\ a].$$

Now

$$f[b\ a\ d_i] = [a\ b\ c],$$

and since $d_i = a$ or b and *all* permutations of $[b\ a\ a]$ and $[b\ a\ b]$ are now available, *all* permutations of $[a\ b\ c]$ can be generated.

These permutation functions can then be applied to the 6 functions above to give by composition the remaining 12 functions which have just 2 values the same. (For example, to obtain $[b\ c\ b]$, apply to $[b\ a\ b]$ the transposition (a, c), i.e. the function $[c\ b\ a]$.)

Finally, the iterates of, e.g., $[b\ c\ c]$ are $[a\ a\ a]$ and $[b\ b\ b]$, and then

$$f[d_i\ d_i\ d_i] = [c\ c\ c].$$

Proof—Part 2. A function of type $[b\ c\ a]$

Such a function is always a δ-function; by hypothesis, it must be fully conjugated.

For functions of this type, it will be convenient to divide the 27 singulary functions into 5 groups, and to give each group a reference symbol for use in the proof.

(κ) The 3 'constant' functions $[a\ a\ a]$, etc.

(λ) The 9 functions $[c\ b\ b]$, $[a\ c\ c]$, $[b\ a\ a]$ and their permutations.

(μ) The 9 functions $[b\ c\ c]$, $[c\ a\ a]$, $[a\ b\ b]$ and their permutations.

(π) The 3 odd permutations of the marks.

(ρ) The 3 even permutations of the marks.

It will be seen as the argument below develops that whenever a function of type $[b\ c\ a]$ is capable of generating *one* function belonging to a given group it must generate *every* function belonging to that group.

Now

$$\theta\ \ = [a\ b\ c],$$

$$\theta^{(1)} = [b\ c\ a],$$

$$\theta^{(2)} = [c\ a\ b],$$

giving group (ρ).

If $\{p_i\}$ is an arbitrary argument set, $[p_i\ p_i'\ p_i'']$, $[p_i'\ p_i''\ p_i]$, $[p_i''\ p_i\ p_i']$ are all cyclic permutations of $[a\ b\ c]$, and so can all be generated.

Let

$$f[p_i\ p_i'\ p_i''] = [X\ Y\ Z],$$

where X, Y, Z are not necessarily all distinct. Then

$$f[p_i'\ p_i''\ p_i] = [Y\ Z\ X]$$

and

$$f[p_i''\ p_i\ p_i'] = [Z\ X\ Y].$$

Now, for some selected sets $\{p_i\}$, $[X\,Y\,Z]$ will be a cyclic permutation of $[a\,b\,c]$. But this cannot happen for *all* such sets $\{p_i\}$, because in that case f, contrary to hypothesis, would *not* be fully conjugated.

Hence from θ and its iterates we can generate at least one *new* singular function, together with its cyclic permutations. $[X\,Y\,Z]$ will from now on be used to refer to this new function, and $\{p_i\}$ to the set which produces it. There are three cases to consider, according to which group of functions $[X\,Y\,Z]$ belongs to.

Case 1. Two of X, Y, Z equal and the third different (group (λ) or (μ)).
Case 2. $[X\,Y\,Z]$ an odd permutation of the marks (group (π)).
Case 3. $X = Y = Z$ (group (κ)).

CASE 1. $Z = Y \neq X$.

(This choice has merely been made for definiteness. Since the cyclic permutations of $[X\,Y\,Z]$ are all available anyway, it clearly does not matter which pair is taken to be equal.)

$$f[p_i\,p_i'\,p_i''] = [X\,Y\,Y],$$

and since $X \neq Y$, either $X = Y'$ or $X = Y''$.

In addition to group (ρ), we can generate immediately

(α) the 3 permutations of $[X\,Y\,Y]$ and, by iteration of these, the 3 permutations of $[X'\,Y'\,Y']$ and the 3 of $[X''\,Y''\,Y'']$,

that is, either the whole of group (λ) or the whole of group (μ).

If $X = Y'$, $[p_i'\,p_i\,p_i]$ is one of the functions (α) for all choices of i, and so has been constructed. Thus we can generate

$$f[p_i'\,p_i\,p_i] = [Y\,X\,X].$$

If, on the other hand, $X = Y''$, $[p_i''\,p_i\,p_i]$ is one of the functions (α) and then

$$f[p_i''\,p_i\,p_i] = [Y\,X\,X].$$

By a similar method, we construct

(β) the 3 permutations each of $[Y\,X\,X]$, $[Y'\,X'\,X']$, $[Y''\,X''\,X'']$.

(α) and (β) together comprise the groups (λ) and (μ).

Consider next $[p_i\,p_i'\,X]$, which is either $[X\,X'\,X]$, $[X'\,X''\,X]$, or $[X''\,X\,X]$. One of these, $[X'\,X''\,X]$, is a function from group (ρ), one is from (λ), and one is from (μ). So $[p_i\,p_i'\,X]$ can be constructed in all circumstances, and so, similarly, can $[p_i\,X'\,p_i'']$. But

$$f[p_i\,p_i'\,X] = [X\,Y\,X'],$$
$$f[p_i\,X'\,p_i''] = [X\,X''\,Y],$$

and so, whether $Y = X''$ or $Y = X'$, we obtain $[X\,X''\,X']$, an odd permutation of the marks, and hence group (π).

Finally, $[p'_i\, p''_i\, X]$ and $[p'_i\, p''_i\, X']$ are always from group (ρ), (λ), or (μ), which allows the generation of

$$f[p'_i\, p''_i\, X] = [Y\, Y\, X'],$$
$$f[p'_i\, p''_i\, X'] = [Y\, Y\, X''].$$

One of these is necessarily $[Y\, Y\, Y]$, giving group (κ).

Each case in this part of the proof will be illustrated by an example showing the generation of the first function in each group. In each example, $n = 4$ and the same set $\{p_i\}$ has been taken.

EXAMPLE 1.

$f[a\, b\, c] = [b\, c\, a]$ and suppose, say, that $f\begin{bmatrix} b & c & a \\ a & b & c \\ c & a & b \\ a & b & c \end{bmatrix} = [b\, a\, a]$ (λ).

Then

$$f\begin{bmatrix} c & b & b \\ b & a & a \\ a & c & c \\ b & a & a \end{bmatrix} = [a\, b\, b] \quad (\mu), \qquad f\begin{bmatrix} b & c & b \\ a & b & b \\ c & a & b \\ a & b & b \end{bmatrix} = [b\, a\, c] \quad (\pi),$$

$$f\begin{bmatrix} c & a & c \\ b & c & c \\ a & b & c \\ b & c & c \end{bmatrix} = [a\, a\, a] \quad (\kappa).$$

CASE 2. $[X\, Y\, Z]$ an odd permutation of $[a\, b\, c]$.

In this case, all 6 permutations of the marks are generated to start with, that is, groups (π) and (ρ).

$$f[p_i\, p'_i\, p''_i] = [X\, Y\, Z],$$

where, since $[X\, Y\, Z]$ is an odd permutation of the marks,

$$X = Y' = Z''.$$

Define $\{r_i\}$ and $\{s_i\}$ so that

(1) $(r_i, s_i) = (X, Y)$ or (Y, X),

(2) $r_i \neq p_i$ and $s_i \neq p'_i$.

Then choose q_i so that p_i, q_i, r_i are all different, and t_i so that p'_i, s_i, t_i are all different. According as

$$(p_i, p'_i) = (Z, Y),\ (Y, X),\ (X, Z),$$

we have, respectively,

$$\begin{array}{lcccc} (r_i, s_i) & = & (Y, X), & (X, Y), & (Y, X), \\ q_i & = & X\ \ , & Z\ \ , & Z\ \ , \\ t_i & = & Z\ , & Z\ , & Y\ . \end{array}$$

Finally, since $t_i = \bar{X}$, it is possible to choose u_i so that X, t_i, u_i are different.

We now show how, in all cases, it is possible to generate a singular function χ having at least 2 values the same. The usual capital-letter notation has been used.

$$f[p_i\, q_i\, r_i] = [X\ Q\ R],$$
$$f[Z\ r_i\ s_i] = [Y\ R\ S] \quad (\text{since } Z' = Y),$$
$$f[p_i'\ s_i\ t_i] = [Y\ S\ T],$$
$$f[X\ t_i\ u_i] = [Z\ T\ U] \quad (\text{since } X' = Z),$$

and these can all be generated, since all singulary functions on the left have been defined to be permutations of the marks.

(1) If $S = X$, the third function has 2 values the same, unless $T = Z$ in which case the fourth function has (at least) 2 values the same. (2) If $S = Y$, either of the middle 2 functions has the property. (3) If $S = Z$ the second function has 2 equal values, unless $R = X$, in which case the first function will serve. Thus, in all cases, a function χ has been derived.

Now suppose first that χ has all 3 values the same, giving group (κ) by iteration.

Define $\{v_i\}$ so that

$$(p_i, v_i) = (Y,Y),\ (X,Z),\ \text{or } (Z,X).$$

This ensures that $[Y\ p_i\ v_i]$ is either a constant function or a permutation of the marks, and hence admissible. It is easily verified that $[Z\ p_i'\ v_i]$ and $[X\ p_i''\ v_i]$ have the same property. Then

$$f[Y\ p_i\ v_i] = [X\ X\ V],$$
$$f[Z\ p_i'\ v_i] = [Y\ Y\ V],$$
$$f[X\ p_i''\ v_i] = [Z\ Z\ V],$$

and 2 of these functions have exactly 2 values the same (one belonging to group (λ) and one to group (μ)), and their permutations and iterations complete these groups and so the whole set of singulary functions.

On the other hand, suppose that χ has exactly 2 values the same. As usual, this makes available either all 9 functions (λ) or all 9 functions (μ).

Define $w_i = v_i'$ or $w_i = v_i''$, the choice being made so that $[Y\ p_i\ w_i]$ is among these 9. It is easily checked that $[Z\ p_i'\ w_i]$ and $[X\ p_i''\ w_i]$ are all among the same 9 functions, permitting the generation of

$$f[Y\ p_i\ w_i] = [X\ X\ W],$$
$$f[Z\ p_i'\ w_i] = [Y\ Y\ W],$$
$$f[X\ p_i''\ w_i] = [Z\ Z\ W].$$

One of these functions has all 3 values the same and another enables the other remaining group of 9 functions to be constructed, so that groups (κ), (λ), and (μ) have all been produced.

EXAMPLE 2.

$$f[a\,b\,c] = [b\,c\,a],\quad f\begin{bmatrix} b & c & a \\ a & b & c \\ c & a & b \\ a & b & c \end{bmatrix} = [c\,b\,a] \quad (\pi).$$

Let

$$f\begin{bmatrix} c \\ b \\ b \\ b \end{bmatrix} = R, \qquad f\begin{bmatrix} b \\ c \\ c \\ c \end{bmatrix} = S, \qquad f\begin{bmatrix} a \\ a \\ b \\ a \end{bmatrix} = T.$$

Then

$$f\begin{bmatrix} b & c & a \\ a & b & c \\ c & b & a \\ a & b & c \end{bmatrix} = [c\,R\,\ldots], \qquad f\begin{bmatrix} a & c & b \\ a & b & c \\ a & b & c \\ a & b & c \end{bmatrix} = [b\,R\,S],$$

$$f\begin{bmatrix} c & b & a \\ b & c & a \\ a & c & b \\ b & c & a \end{bmatrix} = [b\,S\,T], \qquad f\begin{bmatrix} c & a & b \\ c & a & b \\ c & b & a \\ c & a & b \end{bmatrix} = [a\,T\,\ldots],$$

and so when $S = a$ (even if $R = c$), when $S = b$, and when $S = c$ (even if $T = a$), we have a means of generating a function χ having (at least) 2 values the same.

If χ belongs to group (κ),

$$f\begin{bmatrix} b & b & b \\ b & a & c \\ b & c & a \\ b & a & c \end{bmatrix} = [c\,c\,V], \qquad f\begin{bmatrix} a & c & b \\ a & b & c \\ a & a & a \\ a & b & c \end{bmatrix} = [b\,b\,V],$$

$$f\begin{bmatrix} c & a & b \\ c & c & c \\ c & b & a \\ c & c & c \end{bmatrix} = [a\,a\,V],$$

producing both (λ) and (μ).

If χ belongs to group (λ),

$$f\begin{bmatrix} b & b & c \\ b & a & a \\ b & c & b \\ b & a & a \end{bmatrix} = [c\,c\,W], \qquad f\begin{bmatrix} a & c & c \\ a & b & a \\ a & a & b \\ a & b & a \end{bmatrix} = [b\,b\,W],$$

$$f\begin{bmatrix} c & a & c \\ c & c & a \\ c & b & b \\ c & c & a \end{bmatrix} = [a\,a\,W]$$

will give both (κ) and (μ).

If χ belongs to group (μ), replace the last columns on the left b
a, b, c, b, to produce (κ) and (λ).

CASE 3. $X = Y = Z$.

$$f[p_i\, p_i'\, p_i''] = [X\, X\, X],$$

whose iterates are the other constant functions (κ).

Define $\{l_i\}$ so that

(1) $l_i = X$ or X',

(2) either $l_i = p_i$ or $l_i = p_i'$.

(When l_i is not defined uniquely, either choice may be made.)

According as

$$(p_i, p_i') = (X, X'),\ (X', X''),\ (X'', X),$$

we have respectively

$$l_i = X \text{ or } X',\ X',\ X.$$

Let $f(l_i) = L$. Then

$$f[X\ l_i\ \ldots] = [X'\ L\ \ldots],$$
$$f[l_i\ \ p_i'\ \ldots] = [L\ \ X\ \ldots],$$
$$f[p_i\ l_i\ \ldots] = [X\ \ L\ \ldots].$$

The dots denote entries which remain to be filled, but (and this is t
important point) the definitions have been so chosen that every r
on the left can be completed using *only* the 6 singulary functions so f
available, namely the 3 cyclic permutations (ρ) of the marks, i.e.
$[X\, X'\, X'']$, and the 3 constant functions (κ). The dots on the ri
represent the values of the function corresponding to the unspeci
argument sets, but, whether $L = X$, X', or X'', we have generat
either

$$[X'\ X\ \ \ldots]$$

or

$$[X\ \ X''\ \ldots].$$

either case, and whatever the unspecified entry is, this is a new function ψ which is either

(1) a cyclic permutation of $[X''\ X'\ X]$, i.e. a function belonging to group (π), or

(2) a function taking exactly 2 values, i.e. from group (λ) or (μ).

Of course, the cyclic permutations of ψ are also derived.

If ψ is an odd permutation of the marks, define $\{v_i\}$ so that

$$(p_i, v_i) = (X'', X''),\ (X, X'),\ \text{or}\ (X', X).$$

This ensures that $[X''\ p_i\ v_i]$ is either a constant function or a permutation of the marks, as are $[X\ p_i''\ v_i]$ and $[X'\ p_i'\ v_i]$. These produce

$$f[X''\ p_i\ v_i] = [X\ \ X\ V],$$
$$f[X\ \ p_i''\ v_i] = [X'\ X\ V],$$
$$f[X'\ p_i'\ v_i] = [X''\ X\ V],$$

and 2 of these functions have exactly 2 values the same, one being from group (λ) and one from group (μ). These enable the set to be completed.

If, however, ψ has exactly 2 values the same, it can be used first to produce one of the usual groups, either (λ) or (μ), and if we then define $w_i = v_i'$ or $w_i = v_i''$ so that $[X''\ p_i\ w_i]$ belongs to the group available, then so also do $[X\ p_i''\ w_i]$ and $[p_i'\ X'\ w_i]$, and the generation of

$$f[X''\ p_i\ \ w_i] = [X\ \ X\ \ W],$$
$$f[X\ \ p_i''\ \ w_i] = [X'\ X\ \ W],$$
$$f[p_i'\ \ X'\ w_i] = [X\ \ X''\ W]$$

becomes possible. These give the other group (μ) or (λ) since one of $[X\ X\ W]$, $[X'\ X\ W]$, $[X\ X''\ W]$ must belong to (λ) and another to (μ). Unless $W = X$, they will also give the odd permutations (π) as well. But to cover *all* cases, group (π) can be generated by the following process, now that groups (κ), (λ), and (μ) are all accessible.

$$f[X'\ X\ \ l_i] = [X''\ X'\ L],$$
$$f[p_i'\ \ X'\ l_i] = [X\ \ X''\ L],$$
$$f[X\ \ p_i\ \ l_i] = [X'\ \ X\ \ L],$$

and one of these is an odd permutation of the marks.

EXAMPLE 3.

$$f[a\ b\ c] = [b\ c\ a], \qquad f\begin{bmatrix} b & c & a \\ a & b & c \\ c & a & b \\ a & b & c \end{bmatrix} = [a\ a\ a] \quad (\kappa).$$

Let

$$f\begin{bmatrix} b \\ a \\ a \\ a \end{bmatrix} = L.$$

Then

$$f\begin{bmatrix} a & b & c \\ a & a & a \\ a & a & a \\ a & a & a \end{bmatrix} = [b\ L\ \ldots], \qquad f\begin{bmatrix} b & c & a \\ a & b & c \\ a & a & a \\ a & b & c \end{bmatrix} = [L\ a\ \ldots],$$

$$f\begin{bmatrix} b & b & b \\ a & a & a \\ a & a & b \\ a & a & a \end{bmatrix} = [a\ L\ \ldots],$$

and we have obtained either $[a\ c\ \ldots]$ or $[b\ a\ \ldots]$, which is the function.

If ψ belongs to group (π),

$$f\begin{bmatrix} c & b & a \\ c & a & b \\ c & c & c \\ c & a & b \end{bmatrix} = [a\ a\ V], \qquad f\begin{bmatrix} a & a & a \\ a & c & b \\ a & b & c \\ a & c & b \end{bmatrix} = [b\ a\ V],$$

$$f\begin{bmatrix} b & c & a \\ b & b & b \\ b & a & c \\ b & b & b \end{bmatrix} = [c\ a\ V],$$

and this provides both (λ) and (μ).

If ψ belongs to group (λ),

$$f\begin{bmatrix} c & b & b \\ c & a & c \\ c & c & a \\ c & a & c \end{bmatrix} = [a\ a\ W], \qquad f\begin{bmatrix} a & a & b \\ a & c & c \\ a & b & a \\ a & c & c \end{bmatrix} = [b\ a\ W],$$

$$f\begin{bmatrix} c & b & b \\ b & b & c \\ a & b & a \\ b & b & c \end{bmatrix} = [a\ c\ W]$$

will yield (μ), and if ψ is from group (μ) a similar composition with the last columns on the left replaced by c, a, b, a will yield (λ).

Finally,

$$f\begin{bmatrix} b & a & b \\ b & a & a \\ b & a & a \\ b & a & a \end{bmatrix} = [c\ b\ L], \qquad f\begin{bmatrix} c & b & b \\ b & b & a \\ a & b & a \\ b & b & a \end{bmatrix} = [a\ c\ L],$$

$$f\begin{bmatrix} a & b & b \\ a & a & a \\ a & c & a \\ a & a & a \end{bmatrix} = [b\ a\ L],$$

and one of these functions gives the odd permutations (π).

6. Numerical results

The theorem will now be applied to the problem of enumerating the complete generators, under various specifications.

The first problem to be discussed is that of determining how many of the m^{m^n} (n-place) functions are complete generators. Initially, therefore, all generators will be counted separately unless they are identical, no account being taken of the fact that some functions can be transformed into others by mere permutations of the argument set. Under this specification, the total number of complete generators will be denoted by $\mathscr{C}_m(n)$.

Later, some attention will be given to the problem of enumerating the complete generators when permutational variants are *not* counted separately, and with this stipulation the number of essentially distinct complete generators will be denoted by $c_m(n)$. (This question of rearrangements of the argument set which give effectively the same function must be distinguished carefully from the question of permutations of the set of marks, which was under consideration previously in connexion with conjugate functions.)

We shall write

$$N = m^{m^n}, \quad r = 3^{n-1} - 1, \quad s = 2^{n-1} - 1.$$

It is well known (see, for example, (**3**)) that

$$\mathscr{C}_2(n) = 2^{2s} - 2^s,$$

and we shall now derive the formula for $\mathscr{C}_3(n)$.

Functions of type [b a a]

When the values of the function on the repeated marks have been fixed, the values for $3^n - 3 = 3r$ argument sets remain to be specified, and there are 3 choices for each value.

From the total number (3^{3r}) of ways of specifying these values of t
function must be subtracted the total number of functions under w
the subset $\{a, b\}$ is closed and the total number under which the parti
$a|bc$ is invariant, and a correction must then be made for the funct
which have both these properties.

Among the $3r$ argument sets are $2s$ which consist entirely of a and
So the total number of functions under which $\{a, b\}$ is closed is

$$2^{2s}.3^{3r-2s}.$$

Now suppose that $a|bc$ is invariant under f, and consider argum
sets $\{p_i\}$ which include exactly k members equal to $\bar{a}$ and $(n-k)$ mem
equal to a.

If $k = 0$ then $f(a) = b$, and if $k = n$ then $f(\bar{a}) = a$ for all such argu
sets, since $f(b) = f(c) = a$. So there is no latitude of choice for t
values of k.

Take $1 \leqslant k \leqslant n-1$.

For a given value of k in this range, there is a total of $\binom{n}{k} 2^k$ argum
sets, which fall into $\binom{n}{k}$ blocks of 2^k sets, each block consisting of
which are associated under $a|bc$, that is, argument sets with all the
entries occurring in identical positions and all 2^k possible choices of b a
appearing in the remaining k places.

Now, if $a|bc$ is invariant under f, the 2^k values allotted to f in suc
block must be either all a or all $\bar{a}$, that is, they can be specified in $(1+$
ways. Thus, for each value of k with $1 \leqslant k \leqslant n-1$, there are

$$(1+2^{2^k})^{\binom{n}{k}}$$

choices to be made, so that the total number of functions under w
$a|bc$ is invariant is

$$\prod_{k=1}^{n-1} (1+2^{2^k})^{\binom{n}{k}} \equiv \pi_1 \quad \text{(say)}.$$

The above total, however, includes those functions under which the
$\{a, b\}$ is closed, as well as the partition $a|bc$ being invariant, and a sim
argument shows that the total number of these is

$$\prod_{k=1}^{n-1} (1+2^{2^k-1})^{\binom{n}{k}} \equiv \pi_2 \quad \text{(say)}.$$

Therefore, the total number of δ-functions of type $[b\ a\ a]$, i.e. of com
generators of this type, is

$$3^{3r} - 2^{2s}.3^{3r-2s} - \pi_1 + \pi_2,$$

and $[b\ a\ a]$ is the representative of 6 cases

$$[A\ B\ C] = [b\ a\ a],\ [c\ a\ a],\ [b\ c\ b],\ [b\ a\ b],\ [c\ c\ a],\ [c\ c\ b].$$

Functions of type $[b\ c\ a]$

The $3r$ argument sets which exclude the repeated marks may be associated in r blocks of 3 sets each, each block consisting of sets which are derived from one another by the cyclic permutations $g \to g'$, $g \to g''$, i.e. sets such as $\{p_i\}$, $\{p_i'\}$, $\{p_i''\}$. Of the 27 ways of assigning values to such a block, 3 result in the function not being fully conjugated.

So the total number of complete generators of type $[b\ c\ a]$ is

$$3^{3r} - 3^r,$$

and this type $[b\ c\ a]$ is the representative of the 2 cases

$$[A\ B\ C] = [b\ c\ a],\ [c\ a\ b].$$

Hence, when all complete generators are counted separately, their total number, $\mathscr{C}_3(n)$, is

$$6(3^{3r} - 2^{2s}.3^{3r-2s} - \pi_1 + \pi_2) + 2(3^{3r} - 3^r),$$

and so

$$\mathscr{C}_3(n) = 8.3^{3r} - 6.2^{2s}.3^{3r-2s} - 6\prod_{k=1}^{n-1}(1+2^{2^k})^{\binom{n}{k}} + 6\prod_{k=1}^{n-1}(1+2^{2^k-1})^{\binom{n}{k}} - 2.3^r.$$

The total number of binary generators is

$\mathscr{C}_3(2) = 8.3^6 - 6.2^2.3^4 - 6.5^2 + 6.3^2 - 2.3^2 = 3774,$

the value obtained by Martin (4).

$\mathscr{C}_3(3) = 8.3^{24} - 6.2^6.3^{18} - 6.5^3.17^3 + 6.3^3.9^3 - 2.3^8$

$= 2110663244298.$

$\mathscr{C}_3(4) = 8.3^{78} - 6.2^{14}.3^{64} - 6.5^4.17^6.257^4 + 6.3^4.9^6.129^4 - 2.3^{26}$

$= 1{\cdot}3105.10^{38}$ (to 5 significant digits).

$\mathscr{C}_3(5) = 2{\cdot}5834.10^{115}.$

Permutational variants not counted separately

As we found (**9**) for the case $m = 2$, the problem of enumerating those functions which are genuinely different and not merely permutational variants of one another is a rather more complicated combinatorial problem.

The values of $c_m(n)$ for $m = 2$, $n \leqslant 5$ were calculated in the previous paper (**9**), and we shall now look at the function $c_3(n)$ for small values of n.

$c_3(2)$ is easily obtained. With 2 arguments, the only non-trivial permutation is the simple interchange of the arguments, under which commutative functions are unaffected. Non-commutative binary functions, however, occur in pairs which are permutational variants and are therefore not counted separately in $c_3(2)$.

By a similar enumeration to that above, the number of commutative binary generators is

$$8.3^3-6.2.3^2-6.5+6.3-2.3 = 90,$$

a value first obtained by Swift (8). Hence

$$c_3(2) = 90+\tfrac{1}{2}.3684 = 1932.$$

The *frequency* of any function f is the number of occurrences in $\mathscr{C}_m(n)$ of permutational variants which are effectively equivalent to f.

With $m = 3$, $n = 3$, the frequency of f may be

1, if all permutations of the arguments leave the function unaltered.
2, if only the cyclic orderings of the arguments are distinguishable,
3, if 2 arguments (and 2 only) are indistinguishable,
6, if all 3 arguments are distinguishable.

Consider first functions which satisfy

$$f\begin{pmatrix}b\\a\\a\end{pmatrix} = f\begin{pmatrix}a\\b\\a\end{pmatrix} = f\begin{pmatrix}a\\a\\b\end{pmatrix} \quad \text{and the 5 similar relations,}$$

$$f\begin{pmatrix}a\\b\\c\end{pmatrix} = f\begin{pmatrix}b\\c\\a\end{pmatrix} = f\begin{pmatrix}c\\a\\b\end{pmatrix} = G \quad \text{(say),}$$

$$f\begin{pmatrix}a\\c\\b\end{pmatrix} = f\begin{pmatrix}b\\a\\c\end{pmatrix} = f\begin{pmatrix}c\\b\\a\end{pmatrix} = H \quad \text{(say).}$$

If $G \neq H$, the cyclic orderings (x_1, x_2, x_3) and (x_1, x_3, x_2) of the arguments are distinguishable and so the function has frequency 2; if $G = H$, it has frequency 1.

For a function to be of frequency 3, with, say, x_2 and x_3 the indistinguishable arguments, it must satisfy

$$f\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = f\begin{pmatrix}x_1\\x_3\\x_2\end{pmatrix} \text{ whenever } x_2 \neq x_3,$$

so that, of the 24 argument sets (other than $x_1 = x_2 = x_3$), the values at 9 pairs must be the same, the values for the remaining 6 sets being arbitrary. A correction must, of course, be made for the functions included which are of frequency 1, and the result trebled to introduce all 3 transpositions.

The number of functions of frequency 6, which includes the majority of the functions, is then obtained by subtracting from the total number of functions the number with frequency less than 6.

1. When, therefore, $[A\ B\ C]$ has been fixed, but if no other restriction is imposed on the function, the number of functions of frequency 1 is 3^7, of frequency 2 is 3^8-3^7, of frequency 3 is $3(3^{15}-3^7)$, and the rest, namely $3^{24}-3^{16}$ functions, are of frequency 6. For each choice of $[A\ B\ C]$, therefore, the total counted towards $c_3(3)$ is

$$J_1 \equiv 3^7+\tfrac{1}{2}(3^8-3^7)+\tfrac{1}{3}.3(3^{15}-3^7)+\tfrac{1}{6}(3^{24}-3^{16})$$
$$= 47078766054.$$

When allowance is made for the other conditions that the complete generator must satisfy, 4 more calculations of this type are needed, one corresponding to each term in the above formula for $\mathscr{C}_3(3)$. The following summary should give sufficient information for any reader who is interested to be able to check the calculation.

2. $[A\ B\ C] = [b\ a\ a]$ and $\{a,b\}$ closed under f.

$$J_2 \equiv 2^2.3^5+\tfrac{1}{2}(2^2.3^6-2^2.3^5)+\tfrac{1}{3}.3(2^4.3^{11}-2^2.3^5)+\tfrac{1}{6}(2^6.3^{18}-2^4.3^{12})$$
$$= 4133903364.$$

3. $[A\ B\ C] = [b\ a\ a]$ and $a\,|\,bc$ invariant under f.

$$J_3 \equiv 5.9+\tfrac{1}{2}(5.17-5.9)+\tfrac{1}{3}.3(5^2.9.17-5.9)+\tfrac{1}{6}(5^3.17^3-11425)$$
$$= 104295.$$

4. $[A\ B\ C] = [b\ a\ a]$, $\{a,b\}$ closed, *and* $a\,|\,bc$ invariant.

$$J_4 \equiv 3.5+\tfrac{1}{2}(3.9-3.5)+\tfrac{1}{3}.3(3^2.5.9-3.5)+\tfrac{1}{6}(3^3.9^3-1197)$$
$$= 3492.$$

5. $[A\ B\ C] = [b\ c\ a]$ and f not fully conjugated.

$$J_5 \equiv 0+\tfrac{1}{2}.0+\tfrac{1}{3}.3.3^5+\tfrac{1}{6}(3^8-3^6)$$
$$= 1215.$$

Finally,

$$c_3(3) = 8J_1-6J_2-6J_3+6J_4-2J_5$$
$$= 351826101000.$$

With $n = 4$, it is readily discovered that the functions of frequency 24 form such an overwhelming majority that $\frac{1}{24}\mathscr{C}_3(4)$ will give the value of $c_3(4)$ correct to at least 10 and probably many more significant digits.

For, the 2 terms which dominate the value of $\mathscr{C}_3(4)$ are $8.3^{78} - 6.2^{14}$... It is easily checked that of the 8.3^{78} ($= 1{\cdot}3.10^{38}$) functions, the numb... frequency 12 or less is only of the order of $8.6.3^{51}$ ($= 1{\cdot}0.10^{26}$). Simil... of the $6.2^{14}.3^{64}$ ($= 3{\cdot}4.10^{35}$) functions (which are, for example, of t... $[b\ a\ a]$ and leave the set $\{a, b\}$ closed) the number of frequency 1... less is only of the order of $6.6.2^{10}.3^{41}$ ($= 1{\cdot}3.10^{24}$).

8. Asymptotic expressions

The total number of complete generators, $\mathscr{C}_3(n)$, obviously te... asymptotically to

$$8.3^{3^r} = \frac{8}{27} N \quad (N = 3^{3^n}).$$

This can be described informally by saying that of the 27 singul... functions of 3 marks, only 8 are permitted for $[A\ B\ C]$, and for la... values of n the other conditions do not materially affect the enumerat...

By a similar method to that used in (**10**), it is easily shown that c_3... tends asymptotically to

$$\frac{8N}{27n!}.$$

Indeed, the above reasoning makes very plausible the conjecture th... for general m,

$$\mathscr{C}_m(n) \sim \left(\frac{m-1}{m}\right)^m m^{m^n}$$

and

$$c_m(n) \sim \frac{1}{n!}\mathscr{C}_m(n),$$

though no necessary and sufficient conditions for $m > 3$ have yet ... obtained to substantiate this conjecture. It would mean that whe... values of both m and n were reasonably large, the number of com... generators could be taken as e^{-1} times the total number of functions.

The results for $m = 2$ and $m = 3$ are collected in the table below.

9. Conjugate functions not counted separately

If a function is a complete generator, clearly all its $m!$ conjugat... also complete generators, and in the table such functions have ... cases been counted separately. If the members of a mutually conj... set of complete generators are treated as one for purposes of enumer... then all the numerical results in the table must be divided by $m!$.

m		2	3	4	5	Asymptotic expression
2	$\mathscr{C}_2(n)$	1, 2	56	16 256	1 073 709 056	$\frac{1}{4}N$
						$(N = 2^{2^n})$
	$c_2(n)$	1, 2	16	980	9 332 768	$\frac{N}{4n!}$
3	$\mathscr{C}_3(n)$	3 774	2 110 663 244 298	$1{\cdot}3105.10^{38}$	$2{\cdot}5834.10^{115}$	$\frac{8}{27}N$
						$(N = 3^{3^n})$
	$c_3(n)$	1 932	351 826 101 000	$5{\cdot}4603.10^{36}$	$2{\cdot}1528.10^{113}$	$\frac{8N}{27n!}$

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