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TABLES OF THE MODIFIED HANKEL FUNCTIONS OF ORDER ONE-THIRD AND OF THEIR DERIVATIVES

BY

THE STAFF OF THE COMPUTATION LABORATORY

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h_1 & h_2 are modified Hankel fns

Asymptotic expansions.

The asymptotic expansions for $h_1(z)$ and $h_2(z)$ can be used to calculate values beyond the range of the tables, although in general with accuracy less than that of the tabular entries. The asymptotic expansions also give a qualitatively correct account of the behavior of the functions, even over most of the region covered by the tables.

The expansion

$$h_1(z) \sim \alpha z^{-\frac{1}{4}} e^{\frac{2}{3} i z^{\frac{3}{2}} - \frac{5\pi i}{12}} \left[1 + \sum_{m=1} (-i)^m C_m z^{-\frac{3m}{2}}\right] ; \; C_m = \frac{(9-4)(81-4)\cdots(9[2m-1]^2-4)}{2^{4m} 3^m m!} \tag{29}$$

represents $h_1(z)$ for $-\frac{2\pi}{3} < \arg z < \frac{4\pi}{3}$, but cannot be used along the ray $\arg z = -\frac{2\pi}{3}$ (since $h_1(z)$ is a single-valued function, this is, for $h_1(z)$ itself, the same as $\arg z = \frac{4\pi}{3}$). This ray is the branch-cut for the multiple-valued functions that occur in the right-hand member of (12). The numerical coefficient is

$$\alpha = 2^{\frac{1}{3}} 3^{\frac{1}{6}} \pi^{-\frac{1}{2}} = 0.853\,667\,218\,838\,951 \;;$$

we have also

$$W(h_1, h_2) = -2i\alpha^2.$$

An asymptotic expansion valid on the ray $\arg z = -\frac{2\pi}{3}$ is

$$h_1(z) \sim \alpha z^{-\frac{1}{4}} e^{\frac{2}{3} i z^{\frac{3}{2}} - \frac{5\pi i}{12}} \left[1 + \sum_{m=1} (-i)^m C_m z^{-\frac{3m}{2}}\right]$$
$$+ \alpha z^{-\frac{1}{4}} e^{-\frac{2}{3} i z^{\frac{3}{2}} - \frac{11\pi i}{12}} \left[1 + \sum_{m=1} (i)^m C_m z^{-\frac{3m}{2}}\right]. \tag{30}$$

This expansion holds for $-\frac{4\pi}{3} < \arg z < 0$; the branch-cut for the fractional powers of z can be drawn anywhere within the sector $0 < \arg z < \frac{2\pi}{3}$, (refs.: 33; 32; 39, pages 196-220; 4).

The existence of two expressions of different forms which represent asymptotically the same integral function, $h_1(z)$, is an example of Stokes' phenomenon, (refs.: 39, pages 201 - 202; 34; 35; 36; 37; 12; 18). It is indeed immediately obvious that a single expression of the form (29) could not hold for all z, since the right-hand member involves multiple - valued functions, while $h_1(z)$ is single-valued.

There is no contradiction involved in the overlapping of the regions of asymptotic validity of the expansions (29) and (30). For if any particular term of series (29) is chosen, then if $|z|$ is made sufficiently large along any ray lying in the regions of validity of both expansions, the difference in value between (29) and (30) becomes smaller than the term in question. Thus the difference is asymptotically negligible compared with the remainder after taking any number of terms of the series (29). In the region $-\frac{2\pi}{3} < \arg z < 0$, the first line of the right-hand member of (30) is identical with the right-hand member of (29) and the second line is

Coefficients of Hankel functions Type 1

Needs name card

TABLE II

The Coefficients C_m in the Asymptotic Series for $h_1(z)$ and $h_2(z)$

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m	C_m	
1	0.1041 6666 6666 6666 7	
2	0.0835 5034 7222 2222 2	
3	0.1282 2657 4556 3271 6	
4	0.2918 4902 6464 1404 6	
5	0.8816 2726 7443 7576 5	1
6	3.3214 0828 1862 768	3
7	14.9957 6298 6862 6	15
8	78.9230 1301 1587	79
9	474.4515 3886 8	474
10	3207.4900 91	3207
11	2 4086.5496	24087
12	19 8923.12	198923
13	179 1902.0	179 1902
14	1748 4377.	1748 4377

Note: See equations (29), (30), (31), and (32) for the complete asymptotic expansions making use of these coefficients.

TABLE III

Useful Constants

$$k = 1.3103\ 7069\ 7104\ 448 \ - \ 0.7565\ 4287\ 4711\ 451\ i$$

$$\frac{1}{2k} = 0.2861\ 7856\ 0638\ 333 \ + \ 0.1652\ 2526\ 9020\ 841\ i$$

$\alpha = 0.8536\ 6721\ 8838\ 952$ $\qquad W = -1.4574\ 9544\ 1040\ 461\ i$

$\frac{\sqrt{3}}{2} = 0.8660\ 2540\ 3784\ 439$ $\qquad \frac{\sqrt{3}}{3} = 0.5773\ 5026\ 9189\ 626$

$\left(\frac{3}{2}\right)^{\frac{2}{3}} = 1.3103\ 7069\ 7104\ 448$ $\qquad \left(\frac{3}{2}\right)^{\frac{1}{3}} = 1.1447\ 1424\ 2553\ 332$

$$\left(\frac{2}{3}\right)^{\frac{1}{3}} = 0.8735\ 8046\ 4736\ 299$$

$k = \left(\frac{3}{2}\right)^{\frac{2}{3}} \left(1 - \frac{\sqrt{3}}{3} i\right)$ $\qquad \alpha = 2^{\frac{1}{3}}\, 3^{\frac{1}{6}}\, \pi^{-\frac{1}{2}}$

$$W = -2 i \alpha^2$$

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Lucasian Nos. Ref Dickson I p 27

Cunningham,
British Assoc
1894
page 563 – 564

Mersenne nos ~~##~~ $N = 2^{p} - 1$, p prime.
Lucas has shown that N is composite, & contains factor $(2p+1)$ when

- p & $(2p+1)$ are both prime
- and
- p is form $4i+3$

Such nos are called Lucasians

– gives a list
– xerox

BA Report 1894 Dickson I p27

have been assigned. So the order of the group is the product of three numbers, viz., the number of vertices, the number of edges containing a given vertex, and the number of faces passing through a given edge.

4. For the six regular cells of S^4 the results are:

Five-cell	. . .	(C_5) . . .	$5 \times 4 \times 3 = 60$
Eight-cell	. . .	(C_8) . . .	$16 \times 4 \times 3 = 192$
Sixteen-cell	. . .	(C_{16}) . . .	$8 \times 6 \times 4 = 192$
Twenty-four-cell	. .	(C_{24}) . . .	$24 \times 8 \times 3 = 576$
Hundred and twenty-cell		(C_{120}) . . .	$600 \times 4 \times 3 = 7{,}200$
Six hundred-cell	.	(C_{600}) . . .	$120 \times 12 \times 5 = 7{,}200$

5. *Remarks.*—(*a*) A deeper study proves the group of the five-cell to be holohedrically isomorph with that of the icosahedron.

(*b*) In the pairs of cases (C_8, C_{16}) and (C_{120}, C_{600}) the results are equal. This is due to the fact that these pairs of regular cells are reciprocal polars of each other with respect to a hypersphere.

(*c*) The order of the group is equal to $2r$ times the number of faces, r representing the number of vertices situated in any face.

Five-dimensional Space (S^5).

6. *General Principle extended.*—The order of the group is the product of four numbers, viz., the number of vertices, the number of edges through a given vertex, the number of faces through a given edge, and the number of limiting bodies adjacent at a given face.

7. Results:

Six-being	. . .	(B_6) . . .	$6 \times 5 \times 4 \times 3 = 360$
Ten-being	. . .	(B_{10}) . . .	$32 \times 5 \times 4 \times 3 = 1{,}920$
Thirty-two-being	. .	(B_{32}) . . .	$10 \times 8 \times 6 \times 4 = 1{,}920$

8. *Remarks.*—(*a*) The cases (B_{10}) and (B_{32}) are reciprocal polars of each other, &c.

(*b*) The order of the group is equal to $6r$ times the number of limiting bodies, r representing the number of vertices situated in any limiting body.

Space of n-*Dimensions* (S^n).

9. The extension of the principle is evident. The results are:

$$n+1\text{-being } (B_{n+1}) \;..\; (n+1)\,n\,(n-1) \;...\; \times 4 \times 3 = \tfrac{1}{2}(n+1)!$$
$$2n\text{-being } (B_{2n}) \;...\; 2^n \,.\, n\,(n-1) \;....\; \times 4 \times 3 = 2^{n-1} \,.\, n!$$
$$2^n\text{-being } (B_{2^n}) \;...\; 2n \,.\, 2\,(n-1) \;....\; \times 6 \times 4 = 2^{n-1} \,.\, n!$$

10. *Remarks.*—(*a*) The cases (B_{2n}) and (B_{2^n}) are reciprocal polars of each other, &c.

(*b*) The order of the group is equal to $(n-2)!\,r$ times the number of limiting beings of $n-2$ dimensions, r representing the number of vertices situated in each of these.

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8. *On Mersenne's Numbers.* *By* Lieut.-Colonel ALLAN CUNNINGHAM, *R.E., Fellow of King's College, London.*

These are numbers of form $N = 2^p - 1$, where p is prime. Lucas has shown that N is composite, and contains the factor $(2p+1)$ when p and $(2p+1)$ are *both* prime, and p is of form $(4i+3)$.

Such numbers N may for shortness be called *Lucasians.* The highest Lucasians, determinable by the existing tables of primes (extending to 9,000,000), are given by

$$p = 4{,}499{,}591 \text{ and } 4{,}499{,}783,$$

From Lucasian numbers.
Type 3

and these are the only values of p yielding Lucasians in the range of 500 numbers between 4,499,500 and 4,500,000. An interesting group is given by

$$p = 2^3 + 3 = 11\,;\ p = 2^7 + 3 = 131\,;\ p = 2^{15} + 3 = 32{,}771\,;$$

and these are the *only* numbers of form $(2^x + 3)$ yielding Lucasians when x not > 26. Higher values go beyond the tables of primes.

Complete list of primes p of form $(4i + 3)$, with $(2p + 1)$ also prime, when p not $> 2{,}500$; these all give composites for N, and $(2p + 1)$ is a factor of N.

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$p =$ 11, 23, 83, 131, 179, 191, 239, 251, 359, 419, 431, 443, 491, 659, 683, 719, 743, 911, 1,019, 1,031, 1,103, 1,223, 1,439, 1,451, 1,499, 1,511, 1,559, 1,583, 1,811, 1,931, 2,003, 2,039, 2,063, 2,339, 2,351, 2,399, 2,459.

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ignore internal commas!

It seems *probable* that primes of one of forms $p = (2^x \pm 1)$, $(2^x \pm 3)$ will, with exception of those yielding Lucasians, *generally yield prime values* of N, and that *no others will*; all the known (and conjectured) prime Mersenne's numbers fall under this rule.

9. *End Games at Chess.* *By* Lieut.-Colonel ALLAN CUNNINGHAM, *R.E., Fellow of King's College, London.*

Investigation of the number of positions in all the 'end games' at chess when there are only two or three pieces on the board. The results are:—

P = Total number of positions
C = Number of checkmate positions
S = Number of stalemate positions
I = Number of indifferent positions
} with a given set of pieces.

Number of Pieces	Names of Pieces		Number of Positions			
	Black	White	C	S	I	P
2	K	K	0	0	3,612	3,612
3	K	K and Q	324	144	223,476	223,944
3	K	K and R	216	68	223,660	223,944
3	K	K and Kt . . .	0	40	223.904	223,944
3	K	K and B (unnamed) .	0	136	223,808	223,944
3	K	K and WB or BB . .	0	68	111,904	111,972
3	K	K and P (unnamed) .	0	18	195,966	195,984
3	K	K and QRP or KRP .	0	2	24,466	24,468
3	K	K and QKtP or KKtP .	0	3	24,505	24,508
3	K	K and QBP or KBP .	0	3	24,505	24,508
3	K	K and QP or KP . .	0	1	24,507	24,508

DEPARTMENT II.

10. *Experiments showing the Boiling of Water in an open Tube.* *By* Professor OSBORNE REYNOLDS, *F.R.S.*

11. *Report of the Committee on Earth Tremors.*—See Reports, p. 145.

12. *Report of the Committee on Meteorological Photography.*—See Reports, p. 143.

13. *Report of the Committee on Solar Radiation.*—See Reports, p. 106.