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AJM 49 (1927)

AJM 49 (1927) 2494

$5!/12 = 10$. We found above (§ 5 Case 2) $h_3h_2 \cap \mathrm{Cyc}(32) = \mathrm{Cyc}(32) + \mathrm{Cyc}(21^3) + \mathrm{Cyc}(1^5)$. Cyc(32) belongs to a group whose order 6 has the divisors 1, 2, 3, 6. We have $P = 32$, $P^2 = 31^2$, $P^3 = 21^3$, $P^6 = 1^5$; also $A_1 = A_3 = A_6 = 1$, $A_2 = 0$; whence $1^{A_1}2^{A_2}3^{A_3}6^{A_6} = 631$. Thus we obtain the cycle-partition correspondence $32 : 631$. Moreover, since a cycle-partition correspondence between two group operations implies that the same correspondence holds between like powers of those operations, we have at once the correspondences $31^2 : 3^31$, $21^3 : 2^31^4$, and $1^5 : 1^{10}$. Further correspondences are given by $h_3h_2 \cap \mathrm{Cyc}(41) = \mathrm{Cyc}(2^21) + 2\mathrm{Cyc}(1^5)$, namely $41 : 4^22$, and $2^21 : 2^41^2$, and by $h_3h_2 \cap \mathrm{Cyc}(5) = 2\mathrm{Cyc}(1^5)$, namely $5 : 5^2$. Then we have for the complete set of 7 partitions of 5:

Partitions of 5:	5	41	32	31^2	2^21	21^3	1^5
Partitions of 10:	5^2	4^22	631	3^31	2^41^2	2^31^4	1^{10}.

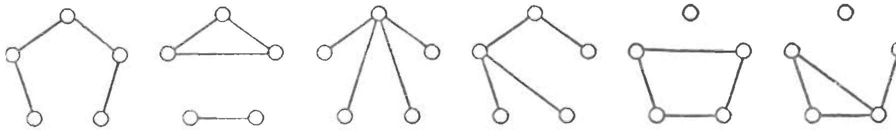
This set of correspondences serves to derive an isomorph of degree 10 from any group of degree 5. Thus from $(1/5)(s_1^5 + 4s_5) = \mathrm{Cyc}(5)$ comes $(1/5)(s_1^{10} + 4s_5^2) = \mathrm{Cyc}(5^2)$. Or taking for G the symmetric group of degree 5, we have from

$$h_5 = (1/120)(s_1^5 + 10s_2s_1^3 + 15s_2^2s_1 + 20s_3s_1^2 + 20s_3s_2 + 30s_4s_1 + 24s_5)$$

the isomorphic G. R. F.

$$h_3h_2 \cup h_5 = (1/120)(s_1^{10} + 10s_2^3s_1^4 + 15s_2^4s_1^2 + 20s_3^3s_1 + 30s_4^2s_2 + 24s_5^2 + 20s_6s_3s_1).$$

The function last written is typical of a class of G. R. F.'s $h_{n-2}h_2 \cup h_n$, of degree $n(n-1)/2$ and order $n!$, connected with the enumeration of the symmetrical aliorelative dyadic "relation-numbers" (Whitehead and Russell, *Principia Mathematica*, Vol. II, p. 301) on a field of n elements. If we represent field members by nodes o, and the holding of a typical relation by connecting lines, then there are 10 possible connecting lines for 5 nodes. If exactly 4 of these 10 are drawn, we get the configurations shown below, which are enumerated by $h_6h_4 \cap (h_3h_2 \cup h_5) = 6$.



(1) (2) (3) (4) (5) (6)

Configurations (5) and (6), which contain isolated nodes, do not, according to the accepted definitions, represent relation-numbers on a field of *five* elements. What we have enumerated are in fact the relation-numbers of a certain type on fields of *five or fewer* elements, and to remove the redundancy we must subtract the corresponding expression for fields of four or fewer elements, thus: $h_6 h_4 \mathbin{\text{♋}} (h_3 h_2 \mathbin{\text{♂}} h_5) - h_2 h_1 \mathbin{\text{♋}} (h_2^2 \mathbin{\text{♂}} h_4) = 6 - 2 = 4$.

By putting every $s_r = 2$ (see § 3 ex. i) in the expressions $h_{n-2} h_2 \mathbin{\text{♂}} h_n$ for various values of n, we find that the total number of symmetrical aliorelative dyadic relation-numbers on fields of n elements is as follows:

Elements in Field:	1	2	3	4	5	6	7	. . .
Relation-Numbers:	0	1	2	7	23	122	888	. . .

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Other species of relation-numbers can be enumerated in a similar manner. Thus for asymmetrical aliorelative dyadic relation-numbers we use the functions $h_{n-2} h_1^2 \mathbin{\text{♂}} h_n$: for symmetrical aliorelative triadic relation numbers the functions $h_{n-3} h_3 \mathbin{\text{♂}} h_n$; for aliorelative triadic relation-numbers symmetrical in two only of their arguments, the functions $h_{n-3} h_2 h_1 \mathbin{\text{♂}} h_n$.

Sometimes the isomorphism is not simple but j-fold; in such case every derived operation will be repeated j times, and this will give a leading term js_1^n inside the parentheses of the G. R. F. When this factor j is carried outside the parentheses, the G. R. F. will appear as of the correct order, so that no exception arises to the rule for finding the expression for $\mathrm{Grf}(H) \mathbin{\text{♂}} \mathrm{Grf}(G)$.

Other isomorphs can be obtained by using as transforming function, instead of a G. R. F. $\mathrm{Grf}(H)$, any symmetric function which is the sum of two or more G. R. F.'s of the proper degree. Thus for enumerating the symmetrical dyadic relation numbers which are not restricted to be aliorelative, we use functions $(h_{n-2} h_2 + h_{n-1} h_1) \mathbin{\text{♂}} h_n$. Such isomorphs are necessarily intransitive. On the other hand, when a single G. R. F. is used as transforming function, the derived isomorph *may* be intransitive, and if it is, still other isomorphs can be got by omitting some of the transitive constituents. The general problem of actually separating an intransitive G. R. F. into transitive constituents has however not been solved, and seems to be closely connected in nature and difficulty with the decomposition of ♋-products into sums of G. R. F.'s.

Some important general relations are (for G and H of degree n, H of order μ): $h_n \mathbin{\text{♂}} \mathrm{Grf}(G) = s_1$ for every G: $(h_n + a_n) \mathbin{\text{♂}} \mathrm{Grf}(G) = (1/2)(s_1^2 + s_2)$ or s_1^2 according as G has or has not odd permutations: $h_{n-1} h_1 \mathbin{\text{♂}} \mathrm{Grf}(G) = \mathrm{Grf}(G)$;

$$h_{(n!/\mu)-1} h_1 \mathbin{\text{♋}} [\mathrm{Grf}(H) \mathbin{\text{♂}} \mathrm{Grf}(G)] = \mathrm{Grf}(H) \mathbin{\text{♋}} \mathrm{Grf}(G).$$