A recursive formula
for even order harmonic series *

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Abstract: A useful recursive formula for obtaining the infinite sums of even order harmonic series $\sum_{n=1}^{\infty} (1/n^{2k})$, $k = 1, 2, \ldots$, is derived by an application of Fourier series expansion of some periodic functions. Since the formula does not contain the Bernoulli numbers, infinite sums of even order harmonic series may be calculated by the formula without the Bernoulli numbers. Infinite sums of a few even order harmonic series, which are calculated using the recursive formula, are tabulated for easy reference.

Keywords: Harmonic series, recursive formula, Fourier series, infinite sum.

1. Introduction

The infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+\epsilon}} \tag{1}$$

is known to converge to a finite value for any positive value of ϵ. When ϵ is a positive integer, the infinite sum of the series (1) can be calculated as [1]:

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+m}} = \frac{(-1)^m}{m!} \Psi^{(m)}(1), \quad m = 1, 2, \ldots, \tag{2}$$

where the psi function Ψ is defined by

$$\Psi(x) = \Gamma'(x)/\Gamma(x), \tag{3}$$

with Γ being the gamma function. In addition, when ϵ is an odd integer (that is, in the case of *even order harmonic series*), the infinite sum of the series (1) can also be calculated by the formula,

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = (-1)^{k-1} \frac{(2\pi)^{2k}}{2(2k)!} B_{2k}, \quad k = 1, 2, \ldots, \tag{4}$$

where B_i, $i = 1, 2, \ldots$, are the Bernoulli numbers and can be calculated by the formula,

$$\sum_{k=0}^{m} {}_{m+1}C_k B_k = 0, \quad m = 1, 2, \ldots, \tag{5}$$

with $B_0 = 1$ [2,3]. More typical values of the Bernoulli numbers may be found in [1].

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As we can see from (4) and (5), in order to find the infinite sum of an even order harmonic series by (4), it is required first to calculate the Bernoulli number.

In this paper, using Fourier series expansion of appropriate periodic functions, a recursive formula for obtaining the infinite sums of even order harmonic series will be derived which does not require the calculation of Bernoulli numbers; that is, the main objective in this paper is to find a formula with which the infinite sums of even order harmonic series can be obtained without an intervening parameter.

2. A recursive formula for even order harmonic series

The Fourier series of a periodic function $g(x)$ may be expressed as

$$
g(x) = A_0 + \sum_{n=1}^{\infty} \left[A_n \cos \frac{2\pi n x}{T} + B_n \sin \frac{2\pi n x}{T} \right],
\tag{6}
$$

where T is the period of $g(x)$, and A_0, A_n and B_n, $n = 1, 2, \ldots$, are called the Fourier coefficients of the function [4,5].

In order to derive the desired formula, let us consider the functions $f_k(x)$, $k = 1, 2, \ldots$, defined by

$$
f_k(x) = (x - \pi)^{2k}, \quad 0 \leqslant x \leqslant \pi,
\tag{7}
$$

with $f_k(-x) = f_k(x)$ and $f_k(x + 2\pi) = f_k(x)$. For the functions $f_k(x)$, $k = 1, 2, \ldots$, it is quite easy to see that the Fourier coefficients $B_n = 0$, $n = 1, 2, \ldots$, and therefore they can be expressed by the following Fourier series,

$$
f_k(x) = a_{2k,0} + \sum_{n=1}^{\infty} a_{2k,n} \cos nx = \sum_{n=0}^{\infty} a_{2k,n} \cos nx,
\tag{8}
$$

where the Fourier coefficients $a_{2k,n}$, $n = 0, 1, \ldots$, (in which the dependence on k is now explicitly denoted) are given by (see Appendix A)

$$
a_{2k,0} = \pi^{2k}/(2k+1)
\tag{9}
$$

and

$$
a_{2k,n} = \frac{4k\pi^{2k-2}}{n^2} - \frac{2k(2k-1)}{n^2} a_{2k-2,n}, \quad n = 1, 2, \ldots .
\tag{10}
$$

After some manipulations as shown in Appendix B, (10) can be transformed into

$$
a_{2k,n} = \frac{2(2k)!\pi^{2k-2}}{n^2} \sum_{j=1}^{k-1} \left(\frac{-1}{n^2\pi^2} \right)^j \frac{1}{(2k-2j-1)!}, \quad n = 1, 2, \ldots .
\tag{11}
$$

Once we have (8), (9) and (11), it is quite straightforward to proceed and to obtain the desired formula. Let us put $x = 0$ in (8), and use (9) and (11) to get

$$
\pi^{2k} = \frac{\pi^{2k}}{2k+1} + \pi^{2k-2} \sum_{n=1}^{\infty} \frac{2(2k)!}{n^2} \sum_{j=1}^{k-1} \left(\frac{-1}{n^2\pi^2} \right)^j \frac{1}{(2k-2j-1)!}.
\tag{12}
$$

After a few steps as shown in Appendix C, (12) becomes

$$C_k = (-1)^{k-1}\pi^{2k}\left[\frac{k}{(2k+1)!} - \sum_{j=0}^{k-2}\frac{1}{(2k-2j-1)!}\left(\frac{-1}{\pi^2}\right)^j\frac{C_{j+1}}{\pi^2}\right], \tag{13}$$

where

$$C_j = \sum_{n=1}^{\infty}\frac{1}{n^{2j}}, \quad j = 1, 2, \ldots, \tag{14}$$

are the infinite sums of even order harmonic series. If we proceed one step further for notational convenience, we finally get the desired formula

$$S_k = (-1)^k\left[\sum_{j=1}^{k-1}\frac{(-1)^{j-1}}{(2k-2j+1)!}S_j - \frac{k}{(2k+1)!}\right], \quad k = 1, 2, \ldots, \tag{15}$$

from (13) after a few rearrangements with the definition

$$S_k = \frac{1}{\pi^{2k}}\sum_{n=1}^{\infty}\frac{1}{n^{2k}} = \frac{C_k}{\pi^{2k}} \tag{16}$$

and

$$S_1 = 1/3!. \tag{17}$$

Some values of $1/S_k$ obtained by the recursive formula (15) and (17) are shown in Table 1.

3. Summary

In summary a recursive formula for obtaining infinite sums of even order harmonic series is derived using Fourier series expansion of periodic functions. While the infinite sums of even order harmonic series can also be calculated by other well-known results, the formula derived in this paper does not require *a priori* knowledge of the Bernoulli numbers. A few values obtained by the formula are shown to illustrate the usefulness of the formula.

Table 1
Some values of $1/S_k$

k	$1/S_k$	k	$1/S_k$
1	6		
2	90	7	$\dfrac{3\cdot 13!}{2^{11}}$
3	945		
4	9450	8	$\dfrac{3\cdot 5\cdot 17!}{2^{14}\cdot 3617}$
5	93555		
6	$\dfrac{15!}{2^{11}\cdot 691}$	9	$\dfrac{7!\cdot 19!}{2^{20}\cdot 3\cdot 5\cdot 43867}$

Appendix A

Calculation of Fourier coefficients

$$a_{2k,0} = \frac{1}{\pi} \int_0^\pi (x - \pi)^{2k} \, \mathrm{d}x = \frac{\pi^{2k}}{2k + 1}, \tag{A.1}$$

$$a_{2k,n} = \frac{2}{\pi} \int_0^\pi (x - \pi)^{2k} \cos nx \, \mathrm{d}x$$

$$= \frac{2}{\pi} \left\{ \left[\frac{(x - \pi)^{2k}}{n} \sin nx \right]_0^\pi - \frac{2k}{n} \int_0^\pi (x - \pi)^{2k-1} \sin nx \, \mathrm{d}x \right\}$$

$$= -\frac{4k}{n\pi} \int_0^\pi (x - \pi)^{2k-1} \sin nx \, \mathrm{d}x$$

$$= -\frac{4k}{n\pi} \left\{ -\left[\frac{(x - \pi)^{2k-1}}{n} \cos nx \right]_0^\pi + \frac{2k-1}{n} \int_0^\pi (x - \pi)^{2k-2} \cos nx \, \mathrm{d}x \right\}$$

$$= \frac{4k\pi^{2k-2}}{n^2} - \frac{2k(2k-1)}{n^2} a_{2k-2,n}. \tag{A.2}$$

Appendix B

Derivation of (11)

Several methods may be used to derive (11) including induction and direct iteration of (10). In this appendix one of the possible derivations is given. Let us first define some quantities for notational convenience: let

$$\alpha_k = 4k/n^2\pi^2 \tag{B.1}$$

and

$$\beta_k = -2k(2k - 1)/n^2\pi^2. \tag{B.2}$$

Then we have

$$b_{2k} = \alpha_k + \beta_k b_{2k-2}, \tag{B.3}$$

where

$$b_{2k} = a_{2k,n}/\pi^{2k}. \tag{B.4}$$

Now by consecutively decreasing the index k in (B.3) by 1 and multiplying appropriate quantities and adding them up, we get

$$b_{2k} = \alpha_k + \beta_k b_{2k-2}$$

$$\beta_k b_{2k-2} = \beta_k \alpha_{k-1} + \beta_{k-1}\beta_k b_{2k-4}$$

$$\vdots$$

$$\beta_3\beta_4 \cdots \beta_k b_4 = \beta_3\beta_4 \cdots \beta_k \alpha_2 + \beta_2\beta_3 \cdots \beta_k b_2$$

$$\overline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}$$

$$b_{2k} = \alpha_k + \alpha_{k-1}\beta_k + \cdots + \beta_2\beta_3 \cdots \beta_k \frac{4}{n^2\pi^2}. \tag{B.5}$$

Now noting that

$$
\beta_k \beta_{k-1} \cdots \beta_{k-j} = \frac{(-2k)(2k-1)}{n^2\pi^2} \frac{(-2k+2)(2k-3)}{n^2\pi^2} \cdots
$$

$$
\frac{(-2k+2j)(2k-2j-1)}{n^2\pi^2}
$$

$$
= \frac{(-2)^{j+1}}{(n\pi)^{2(j+1)}} k(k-1) \cdots (k-j)(2k-1)(2k-3) \cdots (2k-2j-1),
$$

(B.6)

(B.5) can be expressed as

$$
b_{2k} = \frac{4k}{n^2\pi^2} + \frac{4(k-1)}{n^2\pi^2} \frac{(-2k)(2k-1)}{n^2\pi^2} + \cdots
$$

$$
+ \frac{(-2)^{k-1}k(k-1) \cdots 2 \cdot 1 \cdot (2k-1)(2k-3) \cdots 3 \cdot 1}{(n\pi)^{2(k-1)}} \frac{4}{n^2\pi^2}
$$

$$
= \frac{4}{n^2\pi^2} \left[k + \frac{(-2)}{n^2\pi^2} k(k-1)(2k-1) + \left(\frac{-2}{n^2\pi^2}\right)^2 k(k-1)(k-2)(2k-1)(2k-3) \right.
$$

$$
\left. + \cdots + \left(\frac{-2}{n^2\pi^2}\right)^{k-1} k(k-1) \cdots 2 \cdot 1 \cdot (2k-1)(2k-3) \cdots 3 \cdot 1 \right].
$$

(B.7)

Since the $(j+1)$th term in the bracket of (B.7) is

$$
\left(\frac{-2}{n^2\pi^2}\right)^j k(k-1) \cdots (k-j)(2k-1)(2k-3) \cdots (2k-2j-1)
$$

$$
= \frac{(-1)^j \cdot 2}{(n^2\pi^2)^j} k(k-j) \frac{(2k-1)!}{(2k-2j)!},
$$

(B.8)

we finally get

$$
b_{2k} = \frac{4}{n^2\pi^2} \sum_{j=0}^{k-1} \frac{(-1)^j \cdot 2}{(n^2\pi^2)^j} k(k-j) \frac{(2k-1)!}{(2k-2j)!} = \frac{8k(2k-1)!}{n^2\pi^2} \sum_{j=0}^{k-1} \frac{(-1)^j(k-j)}{(n^2\pi^2)^j(2k-2j)!}
$$

$$
= \frac{4k(2k-1)!}{n^2\pi^2} \sum_{j=0}^{k-1} \left(\frac{-1}{n^2\pi^2}\right)^j \frac{1}{(2k-2j-1)!}.
$$

(B.10)

(11) immediately follows from (B.4) and (B.10).

Appendix C

Derivation of (13)

If we divide (12) by π^{2k} and then rearrange the result, we get

$$\frac{2k}{2k+1} = \frac{2(2k)!}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{j=0}^{k-1} \frac{1}{(2k-2j-1)!} \left(\frac{-1}{n^2\pi^2}\right)^j$$

or

$$\frac{k\pi^2}{(2k)!(2k+1)} = \sum_{j=0}^{k-1} \frac{1}{(2k-2j-1)!} \left(\frac{-1}{\pi^2}\right)^j \sum_{n=1}^{\infty} \left(\frac{1}{n^2}\right)^{j+1}. \tag{C.1}$$

Now if we use (14) in (C.1) we get

$$\frac{k}{(2k+1)!} = \sum_{j=0}^{k-1} \frac{1}{(2k-2j-1)!} \left(\frac{-1}{\pi^2}\right)^j \frac{C_{j+1}}{\pi^2}$$

$$= \sum_{j=0}^{k-2} \frac{1}{(2k-2j-1)!} \left(\frac{-1}{\pi^2}\right)^j \frac{C_{j+1}}{\pi^2} + \left(\frac{-1}{\pi^2}\right)^{k-1} \frac{C_k}{\pi^2}. \tag{C.2}$$

After a slight rearrangement we finally get

$$C_k = (-1)^{k-1}\pi^{2k} \left[\frac{k}{(2k+1)!} - \sum_{j=0}^{k-2} \frac{1}{(2k-2j-1)!} \left(\frac{-1}{\pi^2}\right)^j \frac{C_{j+1}}{\pi^2} \right]. \tag{C.3}$$

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