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REPORT OF A COMMITTEE APPOINTED FOR THE PURPOSE OF CARRYING ON THE TABLES CONNECTED WITH THE PELLIAN EQUATION FROM THE POINT WHERE THE WORK WAS LEFT BY DEGEN IN 1817.

[From the *British Association Report*, (1893), pp. 73—120.]

We have, on the Pellian Equation, Degen's tables, the title of which is "Canon Pellianus sive Tabula simplicissimam æquationis celebratissimæ $y^2 = ax^2 + 1$ solutionem pro singulis numeri dati valoribus ab 1 usque ad 1000 in numeris rationalibus iisdemque integris exhibens." Autore Carolo Ferdinando Degen. Hafniæ, apud Gerhardum Bonnierum, MDCCCXVII., 8vo. Introductio, pp. v—xxiv. Tabula I. Solutionem æquationis $y^2 - ax^2 - 1 = 0$ exhibens, pp. 3—106. Tabula II. Solutionem æquationis $y^2 - ax^2 + 1 = 0$, quotiescunque valor ipsius a talem admiserit, exhibens, pp. 109—112.

The mode of calculation is explained in the Introduction, and illustrated by the examples of the numbers 209, 173.

As to the first of these, the entry in Table I. is

	14, 2, 5, 3, (2)
	1, 13, 5, 8, 11
209	3220
	46551

where the first line gives the expression of $\sqrt{209}$ as a continued fraction, viz. we have

$$\sqrt{209} = 14 + \frac{1}{2+}\;\frac{1}{5+}\;\frac{1}{3+}\;\frac{1}{2+}\;\frac{1}{3+}\;\frac{1}{5+}\;\frac{1}{2+}\;\frac{1}{28+}\;\frac{1}{2+}\;\&c.,$$

the denominators being 2, 5, 3, (2), 3, 5, 2, then 28, which is the double of the integer part 14, and then again 2, 5, 3, (2), 3, 5, 2, and so on, the parentheses of the (2) being used to indicate that this is the middle term of the period.

The second row gives auxiliary numbers occurring in the calculation of the first row and having a meaning, as will presently appear. Observe that the 11 which comes under the (2) should also be printed in parentheses (11), but this is not done.

The process for the calculation of the x, y is as follows:

209			
14	1	0	+ 1
2	14	1	− 13
5	29	2	+ 5
3	159	11	− 8
(2)	506	35	+ (11)
3	1171	81	− 8
5	4019	278	+ 5
2	21266	1471	− 13
28	46551	3220	+ 1

viz. writing down as a first column the numbers of the first row, and beginning the second column with 1, 14 (14 the number at the head of the first column), and the third column with 0, 1, we calculate the numbers of the second column, $29 = 2 . 14 + 1$, $159 = 5 . 29 + 14$, $506 = 3 . 159 + 29$, &c., and the numbers of the third column in like manner, $2 = 2 . 1 + 0$, $11 = 5 . 2 + 1$, $35 = 3 . 11 + 2$, &c.; and then writing down as a fourth column the numbers of the second row with the signs +, − alternately, we have a series of equations $y^2 - ax^2 = \pm A$, viz.

$$1^2 - 209 . 0^2 = + 1,$$
$$14^2 - 209 . 1^2 = - 13,$$
$$29^2 - 209 . 2^2 = + 5,$$
$$\vdots$$

the last of them being

$$(46551)^2 - 209 (3220)^2 = + 1,$$

this last corresponding as above to the value + 1, and the numbers 46551 and 3220 being accordingly the y and x given in the fourth and third rows of the table.

As to the second of the foregoing numbers, 173, the only difference is that the period has a double middle term, viz. the entry in the Table I. is

173	13, 6, (1, 1) 1, 4, (13, 13) 190060 2499849

The first row gives the expression of $\sqrt{173}$, viz. that is

$$\sqrt{173} = 13 + \frac{1}{6+}\ \frac{1}{(1)+}\ \frac{1}{(1)+}\ \frac{1}{6+}\ \frac{1}{26+}\ \&c.,$$

the denominators being 6, 1, 1, 6, then 26 (the double of the integer part 13), and then again 6, 1, 1, 6, and so on. In the second row I remark that Degen prints the parentheses (13, 13) for the double middle term.

The process for the calculation of the x, y is similar to that in the former case, viz. we have

173			
13	1	0	+ 1
6	13	1	− 4
(1	79	6	+13
1)	92	7	−13
6	171	13	+ 4
26	1118	85	− 1

where the second and third columns begin 1, 13 and 0, 1 respectively, and the remaining terms are calculated $79 = 6.13 + 1$, $92 = 1.79 + 13$, &c., and $6 = 6.1 + 0$, $7 = 1.6 + 1$, &c.; and then writing down as a fourth column the terms of the second row with the signs +, − alternately, we have

$$1^2 - 173.0^2 = + 1,$$
$$13^2 - 173.1^2 = - 4,$$
$$79^2 - 173.6^2 = + 13,$$
$$\vdots$$

the last equation being

$$(1118)^2 - 173\,(85)^2 = - 1,$$

e term for the last equation being always in a case such as the present one, not +1, but −1. The final numbers 1118, 85 are consequently entered not in Table I., but in Table II., viz. the entry in this table is

173	85 1118

and thence we calculate the numbers y, x of Table I., viz. these are

$$2499849 = 2 \,.\, (1118)^2 + 1,$$
$$190060 = 2 \,.\, 1118 \,.\, 85.$$

Generally Table II. gives for each value of a, comprised therein, values of x, y, such that $y^2 = ax^2 - 1$, and then writing $y_1 = 2y^2 + 1$, $x_1 = 2xy$, we have

$$y_1^2 = (2ax^2 - 1)^2 = 4a^2x^4 - 4ax^2 + 1 = a \,.\, 4x^2 (ax^2 - 1) + 1 = ax_1^2 + 1,$$

so that x_1, y_1 are for the same value of a the values of x, y in Table I.

It is to be remarked that the heading of Table II. is not perfectly accurate, for it purports to give for every value of a, for which a solution exists, a solution of the equation $y^2 = ax^2 - 1$. What it really gives is the solution for each value of a for which the period has a double middle term. But if $a = \alpha^2 + 1$, then obviously we have a solution $y = \alpha$, $x = 1$, and for any such value of a the period has a single middle term, viz. the entry in Table I. is

$\alpha^2 + 1$	α, (2α) 1, 1 2α $2\alpha^2 + 1$.

and we, in fact, have

α^2+1			
α	1	0	+1
(2α)	α	1	−1
2α	$2\alpha^2 + 1$	2α	+1

that is,

$$1^2 - (\alpha^2 + 1)\, 0^2 = +1,$$
$$\alpha^2 - (\alpha^2 + 1)\, 1^2 = -1,$$
$$(2\alpha^2 + 1)^2 - (\alpha^2 + 1)\,(2\alpha)^2 = +1.$$

The foregoing instances of the calculation of x, y in the case of the numbers 209 and 173 suggest a table which may be regarded as an extended form of Degen's tables; viz. such a table, from $a=2$ to $a=99$, is as follows:

SPECIMEN OF EXTENDED FORM OF TABLE IN REGARD TO THE PELLIAN EQUATION.

a		y	x	y^2-ax^2
2	1	1	0	+ 1
	(2)	1	1	− 1
	2	3	2	+ 1
3	1	1	0	+ 1
	(1)	1	1	− 2
	2	2	1	+ 1
5	2	1	0	+ 1
	(4)	2	1	− 1
	4	9	4	+ 1
6	2	1	0	+ 1
	(2)	2	1	− 2
	4	5	2	+ 1
7	2	1	0	+ 1
	1	2	1	− 3
	(1)	3	1	+ 2
	1	5	2	− 3
	4	8	3	+ 1
8	2	1	0	+ 1
	(1)	2	1	− 4
	4	3	1	+ 1
10	3	1	0	+ 1
	(6)	3	1	− 1
	6	19	6	+ 1
11	3	1	0	+ 1
	(3)	3	1	− 2
	6	10	3	+ 1
12	3	1	0	+ 1
	(2)	3	1	− 3
	6	7	2	+ 1

a		y	x	y^2-ax^2
13	3	1	0	+ 1
	1	3	1	− 4
	(1	4	1	+ 3
	1)	7	2	− 3
	1	11	3	+ 4
	6	18	5	− 1
14	3	1	0	+ 1
	1	3	1	− 5
	(2)	4	1	+ 2
	1	11	3	− 5
	6	15	4	+ 1
15	3	1	0	+ 1
	(1)	3	1	− 6
	6	4	1	+ 1
17	4	1	0	+ 1
	(8)	4	1	− 1
	8	33	8	+ 1
18	4	1	0	+ 1
	(4)	4	1	− 2
	8	17	4	+ 1
19	4	1	0	+ 1
	2	4	1	− 3
	1	9	2	+ 5
	(3)	13	3	− 2
	1	48	11	+ 5
	2	61	14	− 3
	8	170	39	+ 1
20	4	1	0	+ 1
	(2)	4	1	− 4
	8	9	2	+ 1

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□ first that is ±1 or ±4 6704, 6705

SPECIMEN OF EXTENDED FORM OF PELLIAN EQUATION TABLE—*continued.*

a		y	x	y^2-ax^2
21	4	1	0	+ 1
	1	4	1	− 5
	1	5	1	+ 4
	(2)	9	2	− 3
	1	23	5	+ 4
	1	32	7	− 5
	8	55	12	+ 1
22	4	1	0	+ 1
	1	4	1	− 6
	2	5	1	+ 3
	(4)	14	3	− 2
	2	61	13	+ 3
	1	136	29	− 6
	8	197	42	+ 1
23	4	1	0	+ 1
	1	4	1	− 7
	(3)	5	1	+ 2
	1	19	4	− 7
	8	24	5	+ 1
24	4	1	0	+ 1
	(1)	4	1	− 8
	8	5	1	+ 1
26	5	1	0	+ 1
	(10)	5	1	− 1
	10	51	10	+ 1
27	5	1	0	+ 1
	(5)	5	1	− 2
	10	26	5	+ 1
28	5	1	0	+ 1
	3	5	1	− 3
	(2)	16	3	+ 4
	3	37	7	− 3
	10	127	24	+ 1
29	5	1	0	+ 1
	2	5	1	− 6
	(1)	11	2	+ 5
	(1)	16	3	− 3
	2	27	5	+ 2
	10	70	13	− 1
30	5	1	0	+ 1
	(2)	5	1	− 5
	10	11	2	+ 1
31	5	1	0	+ 1
	1	5	1	− 6
	1	6	1	+ 5
	3	11	2	− 3
	(5)	39	7	+ 2
	3	206	37	− 3
	1	657	118	+ 5
	1	863	155	− 6
	10	1520	273	+ 1
32	5	1	0	+ 1
	1	5	1	− 7
	(1)	6	1	+ 4
	1	11	2	− 7
	10	17	3	+ 1
33	5	1	0	+ 1
	1	5	1	− 8
	(2)	6	1	+ 3
	1	17	3	− 8
	10	23	4	+ 1
34	5	1	0	+ 1
	1	5	1	− 9
	(4)	6	1	+ 2
	1	29	5	− 9
	10	35	6	+ 1

SPECIMEN OF EXTENDED FORM OF PELLIAN EQUATION TABLE—*continued.*

a		y	x	y^2-ax^2
35	5	1	0	+ 1
	(1)	5	1	− 10
	10	6	1	+ 1
37	6	1	0	+ 1
	(12)	6	1	− 1
	12	73	12	+ 1
38	6	1	0	+ 1
	(6)	6	1	− 2
	12	37	6	+ 1
39	6	1	0	+ 1
	(4)	6	1	− 3
	12	25	4	+ 1
40	6	1	0	+ 1
	(3)	6	1	− 4
	12	19	3	+ 1
41	6	1	0	+ 1
	(2	6	1	− 5
	2)	13	2	+ 5
	12	32	5	− 1
42	6	1	0	+ 1
	(2)	6	1	− 6
	12	13	2	+ 1
43	6	1	0	+ 1
	1	6	1	− 7
	1	7	1	+ 6
	3	13	2	− 3
	1	46	7	+ 9
	(5)	59	9	− 2
	1	341	52	+ 9
	3	400	61	− 3
	1	1541	235	+ 6
	1	1941	296	− 7
	12	3482	531	+ 1
44	6	1	0	+ 1
	1	6	1	− 8
	1	7	1	+ 5
	1	13	2	− 7
	(2)	20	3	+ 4
	1	53	8	− 7
	1	73	11	+ 5
	1	126	19	− 8
	12	199	30	+ 1
45	6	1	0	+ 1
	1	6	1	− 9
	2	7	1	+ 4
	(2)	20	3	− 5
	2	47	7	+ 4
	1	114	17	− 9
	12	161	24	+ 1
46	6	1	0	+ 1
	1	6	1	− 10
	3	7	1	+ 3
	1	27	4	− 7
	1	34	5	+ 6
	2	61	9	− 5
	(6)	156	23	+ 2
	2	997	147	− 5
	1	2150	317	+ 6
	1	3147	464	− 7
	3	5297	781	+ 3
	1	19038	2807	− 10
	12	24335	3588	+ 1
47	6	1	0	+ 1
	1	6	1	− 1
	(5)	7	1	+ 2
	1	41	6	− 11
	12	48	7	+ 1
48	6	1	0	+ 1
	(1)	6	1	− 12
	12	7	1	+ 1

SPECIMEN OF EXTENDED FORM OF PELLIAN EQUATION TABLE—*continued.*

a		y	x	y^2-ax^2
50	7	1	0	+ 1
	(14)	7	1	− 1
	14	99	14	+ 1
51	7	1	0	+ 1
	(7)	7	1	− 2
	14	50	7	+ 1
52	7	1	0	+ 1
	4	7	1	− 3
	1	29	4	+ 9
	(2)	36	5	− 4
	1	101	14	+ 9
	4	137	19	− 3
	14	649	90	+ 1
53	7	1	0	+ 1
	3	7	1	− 4
	(1	22	3	+ 7
	1)	29	4	− 7
	3	51	7	+ 4
	14	182	25	− 1
54	7	1	0	+ 1
	2	7	1	− 5
	1	15	2	+ 9
	(6)	22	3	− 2
	1	147	20	+ 9
	2	169	23	− 5
	14	485	66	+ 1
55	7	1	0	+ 1
	2	7	1	− 6
	(2)	15	2	+ 5
	2	37	5	− 6
	14	89	12	+ 1
56	7	1	0	+ 1
	(2)	7	1	− 7
	14	15	2	+ 1
57	7	1	0	+ 1
	1	7	1	− 8
	1	8	1	+ 7
	(4)	15	2	− 3
	1	68	9	+ 7
	1	83	11	− 8
	14	151	20	+ 1
58	7	1	0	+ 1
	1	7	1	− 9
	1	8	1	+ 6
	(1	15	2	− 7
	1)	23	3	+ 7
	1	38	5	− 6
	1	61	8	+ 9
	14	99	13	− 1
59	7	1	0	+ 1
	1	7	1	− 10
	2	8	1	+ 5
	(7)	23	3	− 2
	2	169	22	+ 5
	1	361	47	− 10
	14	530	69	+ 1
60	7	1	0	+ 1
	1	7	1	− 11
	(2)	8	1	+ 4
	1	23	3	− 11
	14	31	4	+ 1
61	7	1	0	+ 1
	1	7	1	− 12
	4	8	1	+ 3
	3	39	5	− 4
	1	125	16	+ 9
	(2	164	21	− 5
	2)	453	58	+ 5
	1	1070	137	− 9

SPECIMEN OF EXTENDED FORM OF PELLIAN EQUATION TABLE—*continued.*

a		y	x	y^2-ax^2
	3	1523	195	+ 4
	4	5639	722	− 3
	1	24079	3083	+ 12
	14	29718	3805	− 1
62	7	1	0	+ 1
	1	7	1	− 13
	(6)	8	1	+ 2
	1	55	7	− 13
	14	63	8	+ 1
63	7	1	0	+ 1
	(1)	7	1	− 14
	14	8	1	+ 1
65	8	1	0	+ 1
	(16)	8	1	− 1
	16	129	16	+ 1
66	8	1	0	+ 1
	(8)	8	1	− 2
	16	65	8	+ 1
67	8	1	0	+ 1
	5	8	1	− 3
	2	41	5	+ 6
	1	90	11	− 7
	1	131	16	+ 9
	(7)	221	27	− 2
	1	1678	205	+ 9
	1	1899	232	− 7
	2	3577	437	+ 6
	5	9053	1106	− 3
	16	48842	5967	+ 1
68	8	1	0	+ 1
	(4)	8	1	− 4
	16	33	4	+ 1
69	8	1	0	+ 1
	3	8	1	− 5

a		y	x	y^2-ax^2
	3	25	3	+ 4
	1	83	10	− 11
	(4)	108	13	+ 3
	1	515	62	− 11
	3	623	75	+ 4
	3	2384	297	− 5
	16	7775	936	+ 1
70	8	1	0	+ 1
	2	8	1	− 6
	1	17	2	+ 9
	(2)	25	3	− 5
	1	67	8	+ 9
	2	92	11	− 6
	16	251	30	+ 1
71	8	1	0	+ 1
	2	8	1	− 7
	2	17	2	+ 5
	1	42	5	− 11
	(7)	59	7	+ 2
	1	455	54	− 11
	2	514	61	+ 5
	2	1483	176	− 7
	16	3480	413	+ 1
72	8	1	0	+ 1
	(2)	8	1	− 3
	16	17	2	+ 1
73	8	1	0	+ 1
	1	8	1	− 9
	1	9	1	+ 8
	(5	17	2	− 3
	5)	94	11	+ 3
	1	487	57	− 8
	1	581	68	+ 9
	16	1068	125	− 1
74	8	1	0	+ 1
	1	8	1	− 10

Specimen of extended Form of Pellian Equation Table—*continued.*

a		y	x	y^2-ax^2
	(1)	9	1	+ 7
	(1)	17	2	− 7
	1	26	3	+ 10
	16	43	5	− 1
75	8	1	0	+ 1
	1	8	1	− 11
	(1)	9	1	+ 6
	1	17	2	− 11
	16	26	3	+ 1
76	8	1	0	+ 1
	1	8	1	− 12
	2	9	1	+ 5
	1	26	3	− 8
	1	35	4	+ 9
	5	61	7	− 3
	(4)	340	39	+ 4
	5	1421	163	− 3
	1	7445	854	+ 9
	1	8866	1017	− 8
	2	16311	1871	+ 5
	1	41488	4759	− 12
	16	57799	6630	+ 1
77	8	1	0	+ 1
	1	8	1	− 13
	3	9	1	+ 4
	(2)	35	4	− 7
	3	79	9	+ 4
	1	272	31	− 13
	16	351	40	+ 1
78	8	1	0	+ 1
	1	8	1	− 14
	(4)	9	1	+ 3
	1	44	5	− 14
	16	53	6	+ 1
79	8	1	0	+ 1
	1	8	1	− 15
	(7)	9	1	+ 2
	1	71	8	− 15
	16	80	9	+ 1
80	8	1	0	+ 1
	(1)	8	1	− 16
	16	9	1	+ 1
82	9	1	0	+ 1
	(18)	9	1	− 1
	18	163	18	+ 1
83	9	1	0	+ 1
	(9)	9	1	− 2
	18	82	9	+ 1
84	9	1	0	+ 1
	(6)	9	1	− 3
	18	55	6	+ 1
85	9	1	0	+ 1
	4	9	1	− 4
	(1)	37	4	+ 9
	(1)	46	5	− 9
	4	83	9	+ 4
	18	378	41	− 1
86	9	1	0	+ 1
	3	9	1	− 5
	1	28	3	+ 10
	1	37	4	− 7
	1	65	7	+ 11
	(8)	102	11	− 2
	1	881	95	+ 11
	1	983	106	− 7
	1	1864	201	+ 10
	3	2847	307	− 5
	18	10405	1122	+ 1

SPECIMEN OF EXTENDED FORM OF PELLIAN EQUATION TABLE—*continued.*

a		y	x	y^2-ax^2
87	9	1	0	+ 1
	(3)	9	1	− 6
	18	28	3	+ 1
88	9	1	0	+ 1
	2	9	1	− 7
	1	19	2	+ 9
	(1)	28	3	− 8
	1	47	5	+ 9
	2	75	8	− 7
	18	197	21	+ 1
89	9	1	0	+ 1
	2	9	1	− 8
	(3	19	2	+ 5
	3)	66	7	− 5
	2	217	23	+ 8
	18	500	53	− 1
90	9	1	0	+ 1
	(2)	9	1	− 9
	18	19	2	+ 1
91	9	1	0	+ 1
	1	9	1	− 10
	1	10	1	+ 9
	5	19	2	− 3
	(1)	105	11	+ 14
	5	124	13	− 3
	1	725	76	+ 9
	1	849	89	− 10
	18	1574	165	+ 1
92	9	1	0	+ 1
	1	9	1	− 11
	1	10	1	+ 3
	2	19	2	− 7
	(4)	48	5	+ 4
	2	211	22	− 7
	1	470	49	+ 3
	1	681	71	− 11
	18	1151	120	+ 1

a		y	x	y^2-ax^2
93	9	1	0	+ 1
	1	9	1	− 12
	1	10	1	+ 7
	1	19	2	− 11
	4	29	3	+ 4
	(6)	135	14	− 3
	4	839	87	+ 4
	1	3491	362	− 11
	1	4330	449	+ 7
	1	7821	811	− 12
	18	12151	1260	+ 1
94	9	1	0	+ 1
	1	9	1	− 13
	2	10	1	+ 6
	3	29	3	− 5
	1	97	10	+ 9
	1	126	13	− 10
	5	223	23	+ 3
	1	1241	128	− 15
	(8)	1464	151	+ 2
	1	12953	1336	− 15
	5	14417	1487	+ 3
	1	85038	8771	− 10
	1	99455	10258	+ 9
	3	1 84493	19029	− 5
	2	6 52934	67345	+ 6
	1	14 90361	1 53719	− 13
	18	21 43295	2 21064	+ 1
95	9	1	0	+ 1
	1	9	1	− 14
	(2)	10	1	+ 5
	1	29	3	− 14
	18	39	4	+ 1
96	9	1	0	+ 1
	1	9	1	− 15
	(3)	10	1	+ 4
	1	39	4	− 15
	18	49	5	+ 1

SPECIMEN OF EXTENDED FORM OF PELLIAN EQUATION TABLE—*continued.*

a		y	x	y^2-ax^2	a		y	x	y^2-ax^2
97	9	1	0	+ 1	98	9	1	0	+ 1
	1	9	1	− 16		1	9	1	− 17
	5	10	1	+ 3		(8)	10	1	+ 2
	1	59	6	− 11		1	89	9	− 17
	1	69	7	+ 8		18	99	10	+ 1
	(1	128	13	− 9					
	1)	197	20	+ 9	99	9	1	0	+ 1
	1	325	33	− 8		(1)	9	1	− 18
	1	522	53	+ 11		18	10	1	+ 1
	5	847	86	− 3					
	1	4757	483	+ 16					
	18	5604	569	− 1					

The meaning hardly requires explanation; for each number a, we have a series of pairs of increasing numbers, y, x, satisfying a series of equations $y^2 = ax^2 \pm b$; thus

$a = 14$

y	x	$y^2 - ax^2$
1	0	$1 - 14 \cdot 0 = 1$,
3	1	$9 - 14 \cdot 1 = -5$,
4	1	$16 - 14 \cdot 1 = +2$,
11	3	$121 - 14 \cdot 9 = -5$,
15	4	$225 - 14 \cdot 16 = +1$.

The following table, calculated under the superintendence of the Committee, extends from $a = 1001$ to $a = 1500$ (square numbers omitted); it is (with slight typographical variations) nearly but not exactly in the form of Degen's Table I., the chief difference being that for a number a having a double middle term, or of the form $a^2 + 1$ (such number being further distinguished by an asterisk), the x, y entered in the table are the solutions, *not* of the equation $y^2 = ax^2 + 1$, but of the equation $y^2 = ax^2 - 1$. As remarked above, if we have $y^2 = ax^2 - 1$, then writing $y_1 = 2y^2 + 1$ and $x_1 = 2xy$, we obtain $y_1^2 = ax_1^2 + 1$.

Moreover, for each value of a, in the first line, the first term, which is the integer part of $\sqrt{a}$, is separated from the other by a semicolon, and the 1, which is the corresponding first term of the second line, is omitted.

The calculations were made by C. E. Bickmore, M.A., of New College, C his values for x and y have been revised as presently mentioned, but it has assumed that his values for the periods and subsidiary numbers (forming the fir second lines of each division of the table) are accurate; in fact, any error t would cause the resulting values of x and y to be wildly erroneous; but (exc a single instance which was accounted for) the errors in x and y were in ever in a single figure or two or three figures only.

The values of x and y were in every case examined by substitution i equation ($y^2 = ax^2 + 1$, or $y^2 = ax^2 - 1$, as the case may be), which should be sa by them. These verifications were for the most part made by A. Graham, M the Observatory, Cambridge. As already mentioned, some errors were detected, and have been, of course, corrected. The values of x, y given in the table thus sat every case the proper equation $y^2 = ax^2 + 1$, or $y^2 = ax^2 - 1$; on the ground above r to, it is believed that the periods and subsidiary numbers are also accurate.

It may be remarked, in regard to the verification of the equation $y^2 = ax^2$: large values of x and y, it is in practice easier and safer to calculate $ax^2 \pm$: then to compare the square root thereof with the given value of y, than to t calculate the value of y^2.

The Table 1001 to 1500.

1001	31 ; 1, 1, 1, 3, 3, 2, (4) 40, 23, 35, 16, 17, 25, (13)	33532 10 60905
1002	31 ; 1, 1, 1, 8, 2, 1, 1, 1, 3, (10) 41, 22, 39, 7, 23, 31, 26, 33, 17, (6)	65 35248 2068 69247
1003	31 ; 1, 2, (31) 42, 21, (2)	285 9026
1004	31 ; 1, 2, 5, 2, 2, 1, 7, 4, 1, 2, 1, 11, 1, (14) 43, 20, 11, 25, 19, 41, 8, 13, 40, 17, 44, 5, 55, (4)	85 24164 59730 2700 96330 24199
1005	31 ; 1, 2, 2, 1, 5, 15, 1, 2, (12) 44, 19, 20, 39, 11, 4, 41, 21, (5)	930 59568 29501 49761
1006	31 ; 1, 2, 1, 1, 5, 1, 3, 2, 1, 1, 1, 1, 1, 9, 1, 20, 4, 5, 1, 1, 12, 6, 1, (30) 45, 18, 29, 33, 10, 43, 15, 22, 31, 27, 30, 25, 37, 6, 55, 3, 15, 11, 30, 33, 5, 9, 53, (2)	4 45346 14025 55749 21748 141 25267 56378 02146 05455
1007	31 ; 1, 2, (1) 46, 17, (38)	15 476
1008	31 ; 1, (2) 47, (16)	4 127
1009*	31 ; 1, (3, 3) 48, (15, 15)	17 540
1010*	31 ; 1, 3, (1, 1) 49, 14, (31, 31)	41 1303
1011	31 ; 1, 3, 1, (9) 50, 13, 47, (6)	265 8426
1012	31 ; 1, 4, 3, 6, 1, 3, 8, 1, (4) 51, 12, 19, 9, 43, 16, 7, 48, (11)	1013 02110 32226 17399
1013*	31 ; 1, 4, 1, 4, 15, 1, 2, (2, 2) 52, 11, 44, 13, 4, 43, 19, (23, 23)	123 52985 3931 66618
1014	31 ; 1, 5, 2, 1, 1, 1, 1, (20) 53, 10, 23, 30, 29, 25, 38, (3)	1 46266 46 56965
1015	31 ; 1, 6, 10, (2) 54, 9, 6, (29)	11076 3 52871
1016	31 ; 1, (6) 55, (8)	8 255
1017	31 ; 1, 8, 7, 1, 6, 4, 1, 3, 5, 1, 1, (6) 56, 7, 8, 49, 9, 13, 41, 16, 11, 31, 32, (9)	9 09655 84992 290 09322 97217
1018*	31 ; 1, 9, 1, 1, 1, 6, 2, (3, 3) 57, 6, 39, 23, 38, 9, 26, (17, 17)	27 28333 870 50499
1019	31 ; 1, 11, 1, 3, 1, 1, 1, 3, 8, 1, 5, 2, (31) 58, 5, 47, 14, 35, 25, 34, 17, 7, 49, 10, 29, (2)	19 07764 36539 608 99233 21730
1020	31 ; 1, (14) 59, (4)	16 511
1021*	31 ; 1, 20, 3, 6, 1, 3, 2, 1, 1, 12, 5, 4, 15, 1, 2, 1, 4, 1, 1, 2, 1, 1, 1, (5, 5) 60, 3, 20, 9, 44, 15, 23, 27, 36, 5, 12, 15, 4, 45, 17, 41, 12, 33, 29, 20. 33, 25, 36, (11, 11)	98 65001 29666 69564 06909 3152 17280 37258 48825 15030