Phil Mag 35 1944

→ 2195 2196



$e^{n\alpha}$ in the expression representing $\psi$. The values so obtained for the coefficients in (83) above are given by

$$A_1=\frac{c}{\pi}\frac{\sin 2\beta_0-2\beta_0\cos 2\beta_0}{2\beta_0-\sin 2\beta_0} \quad \ldots \quad (88)$$

$$(A_1+A_2)=\frac{2c}{\pi}\frac{\sin 2\beta_0-2\beta_0\cos 2\beta_0}{4\beta_0-\sin 4\beta_0} \quad \ldots \quad (89)$$

$$(A_2+A_3)=(A_1+A_2)\frac{c}{\pi}\frac{\frac{1}{2}\sin 4\beta_0-\sin 2\beta_0}{\frac{1}{3}(6\beta_0-\sin 6\beta_0)} \quad \ldots \quad (90)$$

$$A_n+A_{n+1}=(A_{n-1}+A_n)\frac{c}{\pi}\frac{\frac{1}{n}\sin 2n\beta_0-\sin 2\beta_0}{\frac{1}{n+1}(2(n+1)\beta_0-\sin 2(n+1)\beta_0)} \quad \ldots \quad (91)$$

Of the constants $A'_0$, $A_1'$, etc. $A_0'=-3\eta_0^2$; and the remaining coefficients are to be determined by equating to zero the expressions multiplying each $e^{n\alpha}$ in (87) above. Thus the solutions representing flow into a parallel channel within the boundaries specified by $\eta=\eta_0$ have been found; and the fluid movement represented is unique in accordance with the well-known theorem of Helmholtz [(6)]. The study of these solutions in detail must be deferred to a later opportunity.

*References.*


(1) Sampson, Phil. Trans. 182 A (1891).
(2) Lord Rayleigh, Phil. Mag. (Oct. 1893).
(3) Smoluchowski, *Bull. de L'Acad. des Sc. de Cracovie* (Jan. 1907).
(4) Blasius, *Zeit. der Math.* lviii. (1910).
(5) Harrison, Proc. Camb. Phil. Soc. xix. (1916–19).
(6) Helmholtz, *Abh.* t. 1 (1868).
(7) 'Modern Fluid Dynamics.'


## XXXII. *Coefficients for Numerical Integration with Central Differences.*

By Herbert E. Salzer * †.

[Received August 25, 1943.]

The following table lists the coefficients in the formula for numerical integration employing central differences through the fifty-fifth order. Previous calculations have gone only as far as coefficients of the seventh central difference. The coefficients $M_{2s}$ occur in the following formula:

$$\frac{1}{h}\int_a^{a+nh} f(x)\,dx=[\tfrac{1}{2}f(a)+f(a+h)+\ \ldots\ +f(a+(n-1)h)+\tfrac{1}{2}f(a+nh)]$$
$$+\sum_{s=1}^{m-1} M_{2s}[\mu\delta^{2s-1}f(a+nh)-\mu\delta^{2s-1}f(a)]+nh^{2m}M_{2m}f^{(2m)}(\xi). \quad (1)$$

* Communicated by the Author.

† Mathematical Tables Project, National Bureau of Standards, U.S.A.

2195
2196



The coefficient $M_{2s}$ is identical with $B_{2s}^{(2s)}(s)/(2s)!$ where $B_{2s}^{(2s)}(s)$ is the $(2s)$th Bernoulli polynomial of order $2s$ for argument equal to $s$. Equation (1) is known as the Gauss–Encke formula and also as the second Gaussian summation-formula. The following properties of these coefficients, given in (2)–(7), are all demonstrated in Steffensen :

$$M_{2s+1}=0, \quad \ldots \quad (2)$$

$$M_{2s}=\frac{2}{(2s)!}\sum_{r=0}^{r=s}\frac{D^{2r+1}0^{[2s+1]}}{2^{2r+1}(2r+1)(2r+1)!}, \quad \ldots \quad (3)$$

where $D^{2r+1}0^{[2s+1]}$ is the $(2r+1)$th derivative of the central factorial polynomial

$$x^{[2s+1]}\equiv x(x^2-1/4)(x^2-9/4)\ldots\left[x^2-\frac{(2s-1)^2}{4}\right]$$

for $x=0$.

$$M_{2s}=\frac{2}{(2s)!}\int_0^{\frac{1}{2}}\frac{\Gamma(x+s+\frac{1}{2})}{\Gamma(x-s+\frac{1}{2})}dx, \quad \ldots \quad (4)$$

which is identical with Milne-Thomson's

$$M_{2s}=\frac{1}{(2s)!}\int_0^1(x+s-1)(x+s-2)\ldots(x-s)\,dx, \quad \ldots \quad (4a)$$

$$|M_{2s}|\leqslant\binom{2s}{s}2^{-4s}. \quad \ldots \quad (5)$$

For large $s$,

$$M_{2s}\sim\frac{(-1)^s}{2^{2s-1}s^{\frac{1}{2}}\pi^{3/2}} \quad \ldots \quad (6)$$

(At $s=10$, this formula is accurate to about 1 per cent.).

$$M_{2s}=\frac{1^2}{3!\,2^2}M_{2s-2}-\frac{(1.3)^2}{5!\,2^4}M_{2s-4}+\frac{(1.3.5)^2}{7!\,2^6}M_{2s-6}-\ldots$$
$$+\frac{(-1)^s[1.3\ldots(2s-3)]^2}{(2s-1)!\,2^{2s-2}}M_2+\frac{(-1)^s[1.3\ldots(2s-1)]^2}{(2s-1)!\,(2s+1)2^{2s}}. \quad (7)$$

The first twenty coefficients given below were checked, using their exact values, in the cumulative recursion formula (7). All values were checked both by differencing the ratios $M_{2s}/M_{2s+2}$ and by actual integrations.

*Table of Coefficients.*

| | |
|---|---|
| $M_2$ | $-\frac{1}{12}$. |
| $M_4$ | $+\frac{11}{720}$. |
| $M_6$ | $-\frac{191}{60480}$. |

numerators / denominators = #2195 / #2196

| | |
|---|---|
| $M_8$ | $+\dfrac{2497}{3628800}$. |
| $M_{10}$ | $-\dfrac{14797}{95800320}$. |
| $M_{12}$ | $+\dfrac{924\ 27157}{261\ 53487\ 36000}$. |
| $M_{14}$ | $-\dfrac{367\ 40617}{448\ 34549\ 76000}$. |
| $M_{16}$ | $+\dfrac{6\ 14309\ 43169}{32\ 01186\ 85286\ 40000}$. |
| $M_{18}$ | $-\dfrac{2313\ 39458\ 92303}{51090\ 94217\ 17094\ 40000}$. |
| $M_{20}$ | $+\dfrac{1639\ 96886\ 81447}{152579\ 28431\ 37024\ 00000}$. |
| $M_{22}$ | —·00000 00256 38286 986 |
| $M_{24}$ | ·00000 00061 40295 342 |
| $M_{26}$ | —·00000 00014 75583 052 |
| $M_{28}$ | ·00000 00003 55628 077 |
| $M_{30}$ | —·00000 00000 85924 015 |
| $M_{32}$ | ·00000 00000 20805 605 |
| $M_{34}$ | —·00000 00000 05047 538 |
| $M_{36}$ | ·00000 00000 01226 642 |
| $M_{38}$ | —·00000 00000 00298 549 |
| $M_{40}$ | ·00000 00000 00072 762 |
| $M_{42}$ | —·00000 00000 00017 755 |
| $M_{44}$ | ·00000 00000 00004 338 |
| $M_{46}$ | —·00000 00000 00001 061 |
| $M_{48}$ | ·00000 00000 00000 260 |
| $M_{50}$ | —·00000 00000 00000 064 |
| $M_{52}$ | ·00000 00000 00000 016 |
| $M_{54}$ | —·00000 00000 00000 004 |
| $M_{56}$ | ·00000 00000 00000 001 |

*References.*

D. K. Sen, "Interpolation and Summation Formulas and the Properties of Factorials," Journ. Univ. Bombay, vol. xi. pt. 5, pp. 5-23 (Mar. 1943).
J. F. Steffensen, 'Interpolation,' pp. 109–110, 191 (1927).
L. M. Milne-Thomson, 'Calculus of Finite Differences,' pp. 184–187 (1933).
J. B. Scarborough, 'Numerical Mathematical Analysis,' pp. 124-131, 165-167 (1930).
E. T. Whittaker, G. Robinson, 'Calculus of Observations,' pp. 146-149 (1937).
Bulletin National Research Council, No. 92, p. 28 (1933). (Obvious misprint in $M_6$.)