

$e^{n\alpha}$ in the expression representing ψ . The values so obtained for the coefficients in (83) above are given by

$$A_1 = \frac{c \sin 2\beta_0 - 2\beta_0 \cos 2\beta_0}{\pi (2\beta_0 - \sin 2\beta_0)} \quad \dots \quad (88)$$

$$(A_1 + A_2) = \frac{2c \sin 2\beta_0 - 2\beta_0 \cos 2\beta_0}{\pi (4\beta_0 - \sin 4\beta_0)} \quad \dots \quad (89)$$

$$(A_2 + A_3) = (A_1 + A_2) \frac{c \frac{1}{2} \sin 4\beta_0 - \sin 2\beta_0}{\pi \frac{1}{3} (6\beta_0 - \sin 6\beta_0)} \quad \dots \quad (90)$$

$$A_n + A_{n+1} = (A_{n-1} + A_n) \frac{c}{\pi} \frac{\frac{1}{n} \sin 2n\beta_0 - \sin 2\beta_0}{\frac{1}{n+1} (2(n+1)\beta_0 - \sin 2(n+1)\beta_0)} \quad \dots \quad (91)$$

Of the constants A'_0 , A'_1 , etc. $A'_0 = -3\eta_0^2$; and the remaining coefficients are to be determined by equating to zero the expressions multiplying each $e^{n\alpha}$ in (87) above. Thus the solutions representing flow into a parallel channel within the boundaries specified by $\eta = \eta_0$ have been found; and the fluid movement represented is unique in accordance with the well-known theorem of Helmholtz⁽⁶⁾. The study of these solutions in detail must be deferred to a later opportunity.

References.

- (1) Sampson, Phil. Trans. 182 A (1891).
- (2) Lord Rayleigh, Phil. Mag. (Oct. 1893).
- (3) Smoluchowski, Bull. de L'Acad. des Sc. de Cracovie (Jan. 1907).
- (4) Blasius, Zeit. der Math. lviii. (1910).
- (5) Harrison, Proc. Camb. Phil. Soc. xix. (1916-19).
- (6) Helmholtz, Abh. t. 1 (1868).
- (7) 'Modern Fluid Dynamics.'

The coefficient (2s)th Bernoulli function (1) is known Gaussian sum coefficients, given

where $D^{2s+1}O^{(s)}$ polynomial

for $x=0$.

which is identically

M

| M₂

For large s ,

(At $s=10$, this

$$M_{2s} = \frac{1^2}{3! 2}$$

XXXII. Coefficients for Numerical Integration with Central Differences.

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THE following table lists the coefficients in the formula for numerical integration employing central differences through the fifty-fifth order. Previous calculations have gone only as far as coefficients of the seventh central difference. The coefficients M_{2s} occur in the following formula:

$$\begin{aligned} \frac{1}{h} \int_a^{a+nh} f(x) dx &= [\frac{1}{2}f(a) + f(a+h) + \dots + f(a+(n-1)h) + \frac{1}{2}f(a+nh)] \\ &\quad + \sum_{s=1}^{m-1} M_{2s} [\mu \delta^{2s-1} f(a+nh) - \mu \delta^{2s-1} f(a)] + nh^{2m} M_{2m} f^{(2m)}(\xi). \quad (1) \end{aligned}$$

The first ten exact values, checked both directions,

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The coefficient M_{2s} is identical with $B_{2s}^{(2s)}(s)/(2s)!$ where $B_{2s}^{(2s)}(s)$ is the $(2s)$ th Bernoulli polynomial of order $2s$ for argument equal to s . Equation (1) is known as the Gauss-Encke formula and also as the second Gaussian summation-formula. The following properties of these coefficients, given in (2)-(7), are all demonstrated in Steffensen:

$$M_{2s} = \frac{2}{(2s)!} \sum_{r=0}^{r=s} \frac{D^{2r+1} O^{[2s+1]}}{2^{2r+1} (2r+1) (2r+1)!}, \quad \quad (3)$$

where $D^{2r+1}O^{[2s+1]}$ is the $(2r+1)$ th derivative of the central factorial polynomial

$$x^{[2s+1]} \equiv x(x^2 - 1/4)(x^2 - 9/4) \dots \left[x^2 - \frac{(2s-1)^2}{4} \right]$$

for $x=0$.

$$M_{2s} = \frac{2}{(2s)!} \int_0^{\frac{1}{2}} \frac{\Gamma(x+s+\frac{1}{2})}{\Gamma(x-s+\frac{1}{2})} dx, \quad \dots \quad (4)$$

which is identical with Milne-Thomson's

$$M_{2s} = \frac{1}{(2s)!} \int_0^1 (x+s-1)(x+s-2) \dots (x-s) dx, \quad \dots \quad (4a)$$

$$|M_{2s}| \leq \binom{2s}{s} 2^{-4s}. \quad \dots \dots \dots \dots \dots \dots \dots \dots \quad (5)$$

For large s ,

$$M_{2s} \sim \frac{(-1)^s}{2^{2s-1} s^{\frac{1}{2}} \pi^{3/2}} \quad \dots \quad . \quad . \quad . \quad . \quad . \quad (6)$$

(At $s=10$, this formula is accurate to about 1 per cent.).

$$M_{2s} = \frac{1^2}{3! 2^2} M_{2s-2} - \frac{(1 \cdot 3)^2}{5! 2^4} M_{2s-4} + \frac{(1 \cdot 3 \cdot 5)^2}{7! 2^6} M_{2s-6} - \dots \\ + \frac{(-1)^s [1 \cdot 3 \dots (2s-3)]^2}{(2s-1)! 2^{2s-2}} M_2 + \frac{(-1)^s [1 \cdot 3 \dots (2s-1)]^2}{(2s-1)! (2s+1) 2^{2s}}. \quad (7)$$

The first twenty coefficients given below were checked, using their exact values, in the cumulative recursion formula (7). All values were checked both by differencing the ratios M_{2s}/M_{2s+2} and by actual integrations.

Table of Coefficients.

$$\begin{aligned} M_2 &= -\frac{1}{12}, \\ M_4 &= +\frac{11}{720}, \\ M_6 &= -\frac{191}{60480}. \end{aligned}$$

$$\frac{\text{numerators}}{\text{denominators}} = \frac{\#2195}{\#2196}$$

M_8	$+ \frac{2497}{3628800}$			
M_{10}	$- \frac{14797}{95800320}$			
M_{12}	$+ \frac{924}{261} \frac{27157}{53487} \frac{36000}{}$			
M_{14}	$- \frac{367}{448} \frac{40617}{34549} \frac{76000}{}$			
M_{16}	$+ \frac{6}{32} \frac{14309}{01186} \frac{43169}{85286} \frac{40000}{}$			
M_{18}	$- \frac{2313}{51090} \frac{39458}{94217} \frac{92303}{17094} \frac{40000}{}$			
M_{20}	$+ \frac{1639}{152579} \frac{96886}{28431} \frac{81447}{37024} \frac{00000}{}$			
M_{22}	$- \cdot00000 \quad 00256 \quad 38286 \quad 986$			
M_{24}	$- \cdot00000 \quad 00061 \quad 40295 \quad 342$			
M_{26}	$- \cdot00000 \quad 00014 \quad 75583 \quad 052$			
M_{28}	$- \cdot00000 \quad 00003 \quad 55628 \quad 077$			
M_{30}	$- \cdot00000 \quad 00000 \quad 85924 \quad 015$			
M_{32}	$- \cdot00000 \quad 00000 \quad 20805 \quad 605$			
M_{34}	$- \cdot00000 \quad 00000 \quad 05047 \quad 538$			
M_{36}	$- \cdot00000 \quad 00000 \quad 01226 \quad 642$			
M_{38}	$- \cdot00000 \quad 00000 \quad 00298 \quad 549$			
M_{40}	$- \cdot00000 \quad 00000 \quad 00072 \quad 762$			
M_{42}	$- \cdot00000 \quad 00000 \quad 00017 \quad 755$			
M_{44}	$- \cdot00000 \quad 00000 \quad 00004 \quad 338$			
M_{46}	$- \cdot00000 \quad 00000 \quad 00001 \quad 061$			
M_{48}	$- \cdot00000 \quad 00000 \quad 00000 \quad 260$			
M_{50}	$- \cdot00000 \quad 00000 \quad 00000 \quad 064$			
M_{52}	$- \cdot00000 \quad 00000 \quad 00000 \quad 016$			
M_{54}	$- \cdot00000 \quad 00000 \quad 00000 \quad 004$			
M_{56}	$- \cdot00000 \quad 00000 \quad 00000 \quad 001$			

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XXXIII. *Equilibrium*

THE stresses in a beam of interest in connection with the theory of beams. A certain amount of work has been published, but the lack of an adequate

The principal theoretical

(i.) "Westergaard's theory of proportional to the intensity of the load."

(ii.) A note by the author on the infinite extent resting on a rigid foundation.

(iii.) A paper giving the results corresponding to (ii.).

It is clear that the theories differ in many ways, but no detailed discussion can be given here, but a limited examination of the available literature will indicate the present paper.

The system considered, resting on a rigid foundation, is that of a slab of infinite extent resting on a rigid foundation. The relationship between the theory of concentrated loads and the exact mathematical theory of the form of an infinite beam supported by multiple concentrated loads is discussed in practice.