A COTANGENT ANALOGUE OF CONTINUED FRACTIONS

By D. H. Lehmer

The continued iteration of a rational function \( f(x, y) \) of two variables provides an algorithm for the expression of a real number as a sequence of rational numbers. Thus the function

\[
 f(x_1, f(x_2, f(x_3, \ldots )))
\]

becomes an infinite series for \( f(x, y) = x + y \) and an infinite product for \( f(x, y) = xy \). For \( f(x, y) = x + 1/y \) we obtain the regular continued fraction

\[
 x_1 + \frac{1}{x_2 + \frac{1}{x_3 + \ldots}}
\]

By far the most frequently used function is \( f(x, y) = x + y/c \), which gives the "power series"

\[
 x_1 + \frac{x_2 + \frac{x_3 + \ldots}{c}}{c} = x_1 + \frac{x_2}{c} + \frac{x_3}{c^2} + \ldots,
\]

where the \( x \)'s are the coefficients, used when \( c = 10 \) for the decimal representation of real numbers.\(^1\) The algorithm associated with \( f(x, y) = x(1 - y) \) has been discussed by T. A. Pierce.\(^2\)

This paper is concerned with the case of

\[
 f(x, y) = (xy + 1)/(y - x) = \cot (\text{arc cot } x - \text{arc cot } y),
\]

so that (1) becomes the function

\[
 \cot (\text{arc cot } x_1 - \text{arc cot } x_2 + \text{arc cot } x_3 - \ldots).
\]

This function, despite its aspect, is no more transcendental than a regular continued fraction and both functions have many properties in common. Furthermore, in order to obtain sequences of rational approximations to a real number, we specialize the \( x \)'s to be integers, as in the continued fraction, and consider therefore expressions of the form

\[
 \cot \sum_{n=0}^\infty (-1)^n \text{arc cot } n,
\]

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\(^1\) This use of the function \( x + y/c \) is at least 4000 years old. See Amer. Jour. of Semitic Languages and Literature, vol. 36(1920), No. 4. The Babylonians used \( c = 60 \).

where the \( n_r \) are integers. This expression will be called a "continued cotangent", and we shall use the adjective "finite" or "infinite" according as the series in (2) terminates or not. Although finite and infinite sums of arc cotangents of integers have been considered many times, no systematic treatment of such sums appears to have been given.

**Definition of a regular continued cotangent.** The continued cotangent (2) will be said to be regular if

(a) \( n_r \) is an integer \( \geq 0 \) for \( r \geq 0 \).

(b) If (2) is finite and if \( n_k \) is the last \( n \), then

\[
    n_k > n_{k-1} + n_{k-1} + 1.
\]

In all other cases

\[
    n_r \geq n_{r-1} + n_{r-1} + 1.
\]

The principal value of arc cotangent \( n_r \) is understood. In fact, since \( n_r \) is non-negative,

\[
    0 < \text{arc cot } n_r \leq \frac{1}{2} \pi.
\]

The inequalities (3) and (4) seem at first sight unnatural. They are, however, the analogues of the inequalities

\[
    q_k > 1,
\]

\[
    q_r \geq 1
\]

for the incomplete quotients of the continued fraction

\[
    q_0 + \frac{1}{q_1} + \frac{1}{q_2} + \cdots,
\]

which terminates with \( \cdots + \frac{1}{q_k} \) or is infinite. The reason for insisting on the stronger inequality (3) in the case of a finite continued cotangent is the same as the reason for (3') in the continued fraction: to insure for every rational number a unique expansion. As a matter of fact, if (4) held for \( n_k \) but not (3), so that

\[
    n_k = n_{k-1}^2 + n_{k-1} + 1,
\]

then the last two terms of (2) could be replaced by a single term, since

\[
    \text{arc cot } n_{k-1} = \text{arc cot } (n_{k-1}^2 + n_{k-1} + 1) = \text{arc cot } (n_{k-1} + 1),
\]

just as in continued fractions we write

\[
    \frac{1}{|q_{k-1}|} + \frac{1}{1} = \frac{1}{|q_{k-1} + 1|}.
\]

\(^3\text{As in continued fractions we might allow } n_0 \text{ to be negative. However, this extra generality is non-essential for our purposes.}\)
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Hence (3) may as well be assumed. It is perhaps worth noting that this con-
traction of the last two terms cannot be repeated in the continued cotangent
any more than in continued fractions. In fact we would need to have as a
counterpart of (5)

\[ n_{k-1} + 1 = n_{k-2}^2 + n_{k-2} + 1. \]

This violates (4).

**Theorem 1.** Every infinite regular continued cotangent converges.

**Proof.** We need merely to note that

\[ \arccot n_0 - \arccot n_1 + \arccot n_2 - \cdots \]

form an alternating series of terms monotonically decreasing in absolute value in
view of (4). Since (6) converges to a positive quantity, the cotangent of (6)
exists, and this proves the theorem. In fact, it is easy to see that (6) converges
not only absolutely but with tremendous rapidity, more rapidly, indeed, than the
series

\[ \frac{1}{2} + \frac{1}{4} + \frac{1}{16} + \frac{1}{256} + \frac{1}{65536} + \cdots + \frac{1}{2^n} + \cdots, \]

in view of (4) and the inequality

\[ \arccot n_r < \frac{1}{n_r}. \]

This rapidity of convergence is a feature of the continued cotangent not enjoyed
by the continued fraction. The least rapidly converging continued fraction may
be said to be

\[ \frac{\sqrt{5} - 1}{2} = 0 + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \cdots, \]

whereas the least rapidly converging continued cotangent is

\[ \xi = \cot (\arccot 0 - \arccot 1 + \arccot 3 - \arccot 13 + \arccot 183 - \cdots - \arccot 33673 + \arccot 1133904603 - \cdots), \]

in which

\[ n_{k+1} = n_k^2 + n_k + 1. \]

**Uniqueness theorem.** Theorem 1 guarantees that every continued cotangent
represents a real positive number. Before treating the inverse problem of
finding the continued cotangent expansion of a given number, we prove the
following uniqueness theorem.

**Theorem 2.** Two regular continued cotangents can be equal only if they are
identically equal.
Proof. Let
\[
\cot \sum_{r=0} \cdot (-1)^r \cot n_r = \cot \sum_{s=0} \cdot (-1)^s \cot m_s
\]
be two equal regular continued cotangents, and suppose, if possible, that
\( n_r = m_r \) does not hold for all \( r \). Then there exists a first instance, \( r = \tau \), where
\( n_\tau \neq m_\tau \), while \( n_r = m_r \) for \( r < \tau \) if \( \tau \neq 0 \). Then from (9) we have
\[
\sum_{\lambda=0} \cdot (-1)^{\lambda} \cot n_{r+\lambda} = \sum_{\lambda=0} \cdot (-1)^{\lambda} \cot m_{r+\lambda} = S.
\]
Since \( n_r \neq m_r \), at least one of these sums contains two or more terms. Let
this sum be the left one, so that
\[
S = \sum_{\lambda=0} \cdot (-1)^{\lambda} \cot n_{r+\lambda} \geq \cot n_r - \cot n_{r+1}
\]
\[
= \cot \left( n_r + \frac{n_r^2 + 1}{n_{r+1} - n_r} \right) \geq \cot (n_r + 1).
\]
In fact, the first \( \geq \) sign reads = only if \( \cot n_{r+1} \) is the last term of the left
member of (10). In this case, however, (3) applies, so that
\[
n_{r+1} > n_r^2 + n_r + 1, \quad \text{or} \quad \frac{n_r^2 + 1}{n_{r+1} - n_r} < 1.
\]
Therefore the second \( \geq \) sign in (11) reads \( > \) in case the first reads \( = \). That is,
\[
S > \cot (n_r + 1).
\]
But since the left member of (10) contains at least two terms,
\[
S < \cot n_r.
\]
We may now show that the right member of (10) contains two or more terms; otherwise we could write from (10), (12) and (13)
\[
\cot (n_r + 1) < S = \cot m_r < \cot n_r.
\]
That is, \( n_r + 1 > m_r > n_r \). But this is impossible, since these letters are
integers. We conclude, therefore, that both members of (10) contain two or
more terms. Hence not only is \( S < \cot m_r \), so that
\[
m_r < n_r + 1,
\]
but also, since the reasoning used to establish (12) may be now applied to the
\( m \)'s, \( S > \cot (m_r + 1) \). Combining this with (13), we have \( n_r < m_r + 1 \).
Finally in view of (14) we may write \( n_r - 1 < m_r < n_r + 1 \). But this con-
tradicts \( m_r \neq n_r \). Hence the theorem is proved.

\[\]
\[ \sum_{r=0}^{\infty} (-1)^r \text{arc cot } m_r, \]

are cot \( m_{r+1} = S. \n\]

contains two or more terms. Let

\[ n_r + \frac{n_r^2 + 1}{n_{r+1} - n_r} \]

arc cot \( n_r + 1. \n\]

arc cot \( n_{r+1} \) is the last term of the left applies, so that

\[ n_r^2 + 1 < n_{r+1} - n_r < 1. \]

in case the first reads =. That is, \( + 1). \n\]

at least two terms, \( n_r. \n\]

(10) contains two or more terms; (13)

\[ \cot m_r < \text{arc cot } n_r. \]

impossible, since these letters are members of (10) contain two or \( m_r, \) so that

\[ 1. \n\]

(12) may be now applied to the with (13), we have \( n_r < m_r + 1. \n\]

\[ n_r < n_r + 1. \] But this convvved.

tion of the regular continued cotangent

\[ \text{Arc cotangent algorithm.} \quad \text{We now describe an algorithm, analogous to that of Euclid, for generating from a given real positive number its regular cotangent expansion.} \n\]

Let \( x \) be the given positive number. We define two sets of numbers \( x_r \)

and \( n_r \) \((r = 0, 1, 2, \ldots)\) called respectively the \( r \)-th complete and incomplete cotangent of \( x \) as follows:

\[ x_0 = x, \quad n_0 = [x_0], \]

\[ x_1 = \frac{x_0 n_0 + 1}{x_0 - n_0}, \quad n_1 = [x_1], \]

\[ x_2 = \frac{x_1 n_1 + 1}{x_1 - n_1}, \quad n_2 = [x_2], \]

\[ \vdots \]

\[ x_{r+1} = \frac{x_r n_r + 1}{x_r - n_r}, \quad n_{r+1} = [x_{r+1}]. \]

This algorithm is to be continued as long as \( x_{r+1} \) exists, that is, as long as \( x \),

is not an integer \( n_r = [x_r]. \) We next prove

\[ \text{Theorem 3.} \quad \text{The cotangent} \]

\[ \sum_{r=0}^{\infty} (-1)^r \text{arc cot } m_r, \]

where the sum extends over all the intermediate cotangents \( n_r \) of \( x, \) is regular.

\[ \text{Proof.} \quad \text{Obviously (a) is satisfied. To show that (b) is satisfied we set} \]

\( x_r = n_r + e_r, \) where \( 0 < e_r < 1. \) Then (15) becomes

\[ x_{r+1} = \frac{n_r^2 + 1}{e_r} + n_r > n_r^2 + n_r + 1. \]

Hence

\[ [x_{r+1}] = n_{r+1} \geq n_r^2 + n_r + 1, \]

so that (4) is satisfied for \( r \neq k - 1. \) For \( r = k - 1 \) we have from (17)

\[ x_k = n_k > n_{k-1}^2 + n_{k-1} + 1, \]

which is (3). Hence the theorem is true.

\[ \text{Theorem 4.} \quad \text{If } n_0, n_1, n_2, \ldots \text{ are generated by } x, \text{ then} \]

\[ \sum_{r=0}^{\infty} (-1)^r \text{arc cot } n_r = \text{arc cot } x - (-1)^x \text{arc cot } x. \]

\[ \text{Remark.} \quad \text{This theorem justifies the name "complete cotangent" for } x. \]

\[ \text{Proof.} \quad \text{Since} \]

\[ x_{r+1} = \frac{x_r n_r + 1}{x_r - n_r}, \]

Here, as usual, \( [x] \) means the greatest integer \( \leq x. \)
we have

\((-1)^\nu \arccot n_\nu = (-1)^\nu (\arccot x_{\nu+1} + \arccot x_\nu)\).

Setting \(\nu = 0, 1, 2, \ldots, \mu - 1\) and adding, we get the theorem.

Theorem 5.

\(x = \cot \sum_{\nu=0}^{\mu-1} (-1)^\nu \arccot n_\nu,\)

where the sum extends over all incomplete cotangents \(n_\nu\) generated by \(x\).

Proof. In case there exists only a finite number of \(n_\nu\)'s, the last being \(n_\mu\), we may set \(\mu = k\) in (18) and transpose the term \((-1)^k \arccot x_k\). Taking the cotangent of both sides we obtain (19).

In case an infinite number of \(n_\nu\)'s are generated by \(x\) we can write in view of (18) and (4),

\(\lim_{\mu \to \infty} \sum_{\nu=0}^{\mu-1} (-1)^\nu \arccot n_\nu = \arccot x - \lim_{\mu \to \infty} (-1)^\nu \arccot x_\mu = \arccot x.\)

Hence in this case also

\(x = \sum_{r=0}^{\infty} (-1)^r \arccot n_r.\)

Theorem 6. Every positive number has a unique regular continued cotangent expansion.

Proof. The existence of such an expansion follows from the arc cotangent algorithm and Theorem 5, while the uniqueness is provided by Theorem 2.

Theorem 7. The number \(x\) is rational or irrational according as its continued cotangent expansion (19) is finite or not.

Proof. If (19) is finite, it follows from the addition theorem of the cotangent function that \(x\) is rational. This may be seen otherwise. In fact, if \(x\) were irrational, so also would be \(x_1, x_2, \ldots\). Hence there could not exist a \(k\) for which \(x_k\) is an integer to terminate the algorithm.

If (19) is infinite, then \(x\) is irrational. In fact, suppose that \(x = p/q\), where \(p\) and \(q\) are integers. It follows that \(x_r = p_r/q_r\) is also rational for every \(r\). From (15)

\(x_{r+1} = \frac{p_{r+1}}{q_{r+1}} = \frac{p_r n_r + q_r}{p_r - n_r q_r} = \frac{p_r n_r + q_r}{r},\)

where, since \(n_r = [x_r] = [p_r/q_r]\), the denominator \(r\) is the remainder on division of \(p_r\) by \(q_r\), so that \(r < q_r\). Since we may suppose that the fraction \(p_{r+1}/q_{r+1}\) is in its lowest terms, we have the inequality

\(q_{r+1} \leq r < q_r,\)

for every \(r\). But this implies the existence of an infinite sequence \(q_1, q_2, \ldots\) of strictly decreasing positive integers, and this is absurd. Hence \(x\) is irrational.
If \( x \) is a rational number \( p/q \), the successive numerators \( p_n \) and the denominators \( q_n \) of \( x \), can be found as in the greatest common divisor process as follows:

\[
\begin{align*}
p_0 &= n_0 q_0 \quad (0 \leq q_0 < q), & \quad p_0 + q = p_1, \\
p_1 &= n_1 q_0 + q_1 \quad (0 \leq q_1 < q_0), & \quad p_1 + q_1 = p_2, \\
p_2 &= n_2 q_1 + q_2 \quad (0 \leq q_2 < q_1), & \quad p_2 q_2 + q_2 = p_3, \\
& \quad \vdots \\
p_r &= n_r q_r + q_{r+1} \quad (0 \leq q_r < q_{r+1}), & \quad p_r n_r + q_r = p_{r+1}, \\
p_{r+1} &= n_{r+1} q_r. 
\end{align*}
\]

In general, \( p_r \) will not be prime to \( q_r \). In fact, any factor which they may have in common will be a common factor of \( p_{r+1} \) and \( q_{r+1} \) and hence of all further \( p \)'s and \( q \)'s. For example, for \( x = 65/37 \), we find the following values of \( p_r \), \( q_r \), \( n_r \), and the greatest common divisor \( \delta \), of \( p_r \) and \( q_r \):

\[
\begin{array}{cccc}
\nu & 0 & 1 & 2 \\
p_r & 65 & 102 & 334 & 6030 \\
q_r & 37 & 28 & 18 & 10 \\
n_r & 1 & 3 & 18 & 603 \\
\delta & 1 & 2 & 2 & 10 \\
\end{array}
\]

Hence \( 65/37 = \cot(\text{arc cot } 1 - \text{arc cot } 3 + \text{arc cot } 18 - \text{arc cot } 603) \).

Convergents. Let \( n_0, n_1, n_2, \ldots \) be the incomplete cotangents generated by \( x \). Then the curate expansion of \( \mu \) terms

\[
\sigma_{\mu}(x) = \cot \sum_{\nu=0}^{\mu-1} (-1)^\nu \text{arc cot } n_\nu,
\]

is called the \( \mu \)-th convergent of \( x \). It is clearly a rational number depending only on \( \mu \) and \( x \). The following expression relates \( x, \sigma_{\mu}(x) \), and the complete cotangent \( x_\mu \) by (18):

\[
(20) \quad \sigma_{\mu}(x) = \cot (\text{arc cot } x - (-1)^\mu \text{arc cot } x_\mu) = \frac{(-1)^\mu x_\mu x + 1}{(-1)^\mu x_\mu - x}.
\]

Theorem 8. If the integers \( A_0 \) and \( B_0 \) are defined by

\[
A_0 = 1, \quad A_{r+1} = A_r n_r + (-1)^r B_r, \\
B_0 = 0, \quad B_{r+1} = B_r n_r + (-1)^r A_r,
\]

then the \( \mu \)-th complete cotangent is given by

\[
(22) \quad x_\mu = (-1)^\mu A_\mu x + B_\mu, \\
A_\mu - B_\mu x
\]

and the \( \mu \)-th convergent \( \sigma_{\mu}(x) \) is given by

\[
(23) \quad \sigma_{\mu}(x) = A_\mu/B_\mu.
\]
Proof. Formula (22) is easily established by induction. In fact (22) holds for \(\mu = 0\), since \(A_0 = 1, B_0 = 0, x_0 = x\). If it is true for \(\mu = \nu\), we may write by (15) and (21)

\[
x_{\nu+1} = \frac{(-1)^\nu(A, x + B, x) n_\nu + A, x - B, x}{(-1)^\nu(A, x + B, x) - n_\nu(A, x - B, x)}
\]

\[
= (-1)^{\nu+1} \frac{(A, n_\nu - (-1)^\nu B, x + B, n_\nu + (-1)^\nu A, x)}{A, n_\nu - (-1)^\nu B, x - (B, n_\nu + (-1)^\nu A, x)}
\]

so that the induction is complete. Having established (22), we see that (23) follows from (20). In fact,

\[
\sigma_\mu(x) = \frac{A_\mu x^2 + B_\mu x}{A_\mu x - B_\mu x + 1} = A_\mu / B_\mu.
\]

The numbers \(A_\mu\) and \(B_\mu\) are, of course, the analogues of the numerator and denominator of the \(\mu\)-th convergent of the regular continued fraction. However, the recurrence formulas (21) are of a different nature, \(A_\mu\) or \(B_\mu\) depending not on the preceding \(A\)'s or \(B\)'s, but on the preceding \(A\) and \(B\). This fact allows one to give an explicit formula for \(A_\mu\) and \(B_\mu\) in terms of the first \(\mu\) incomplete cotangents \(n_3, n_1, \ldots, n_{\mu-1}\).

\[
\begin{align*}
A_0 &= 1, & B_0 &= 0, \\
A_1 &= n_0, & B_1 &= 1, \\
A_2 &= n_0 n_1 + 1, & B_2 &= n_1 - n_0, \\
A_3 &= n_0 n_1 n_2 + n_0 - n_1 + n_2, & B_3 &= n_0 n_1 - n_0 n_2 + n_1 n_2 + 1, \\
A_4 &= n_0 n_1 n_2 n_3 + n_0 n_1 + n_1 n_2 + n_1 n_3 + n_2 n_3 - n_1 n_2 - n_0 n_3 + 1, \\
B_4 &= n_0 n_1 n_2 n_3 + n_1 n_2 n_3 - n_0 n_2 n_3 + n_2 + n_3 - n_1 - n_0 \cdot n_2.
\end{align*}
\]

The general formula for the \(A_\mu\) and the \(B_\mu\) is given by

**Theorem 9.**

\[
A_\mu + iB_\mu = (n_0 + i)(n_1 - i)(n_2 + i)(n_3 - i) \cdots (n_{\mu-1} + (-1)^{\mu-1}i)
\]

\[
= \prod_{\nu=0}^{\mu-1} (n_\nu + (-1)^\nu i)
\]

\[(i^2 = -1).
\]

In other words, if \(S_\nu\) denotes the sum of the products of \((-1)^\nu n_\nu\) taken \(\nu\) at a time, then for \(\mu > 0\)

\[
\begin{align*}
(-1)^{|\nu|} A_\mu &= S_\mu - S_{\mu-2} + S_{\mu-4} - \cdots, \\
(-1)^{|\nu|} B_\mu &= S_{\mu-1} - S_{\mu-3} + S_{\mu-5} - \cdots.
\end{align*}
\]
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Proof. Formula (24) is easily established by induction, if we use (21). It also follows readily from

\[ \text{arc cot } u = \frac{1}{2i} \log \frac{u + i}{u - i}. \]

**Theorem 10.**

(25) \[ \left| \begin{array}{cc} A_\mu & B_\mu \\ A_{\mu+1} & B_{\mu+1} \end{array} \right| = (-1)^{\mu+1} (A_\mu^2 + B_\mu^2) = (-1)^{\mu+1} \prod_{n=0}^{\mu-1} (n_n^2 + 1). \]

Proof. The first equality follows at once from (21) while the second equality is obtained by taking the squares of the absolute values of both sides of (24).

**Theorem 11.**

(26) \[ A_\mu A_{\mu+1} + B_\mu B_{\mu+1} = \left| \begin{array}{cc} A_\mu & iB_\mu \\ iB_{\mu+1} & A_{\mu+1} \end{array} \right| = n_\mu (A_\mu^2 + B_\mu^2) = n_\mu \prod_{n=0}^{\mu-1} (n_n^2 + 1). \]

Proof. The theorem follows at once from (21) and (25).

For example, the values of \( A, B \) for \( x = 65/37 \) are given in the following table.

\[
\begin{array}{c|cccc}
\nu & 0 & 1 & 2 & 3 & 4 \\
\hline
n_\nu & 1 & 3 & 18 & 603 & \\
A_\nu & 1 & 1 & 4 & 70 & 42250 \\
B_\nu & 0 & 1 & 2 & 40 & 24050 \\
\end{array}
\]

Here we find that \( A_4/B_4 = 65/37 \) and that \( A_4 \) and \( B_4 \) have the common factor \( 650 = (n_3^2 + 1)(n_2^2 + 1) \).

As a second example, we give the elements for \( x = 6954069/2559142 \).

\[
\begin{array}{c|cccc}
\nu & 0 & 1 & 2 & 3 & 4 \\
\hline
p_\nu & 6954069 & 16467280 & 133574025 & 9886258830 & \\
q_\nu & 2559142 & 1835785 & 1781000 & 1780025 & \\
n_\nu & 2 & 8 & 74 & 5554 & \\
A_\nu & 1 & 2 & 17 & 1252 & 6054069 \\
B_\nu & 0 & 1 & 6 & 461 & 2559142 \\
\end{array}
\]

In this example \( A \) and \( B \) have no common factor. It is clear from (25) that any factor common to \( A \) and \( B \) will divide \((n_3^2 + 1)(n_2^2 + 1) \ldots (n_0^2 + 1)\), and this factor will also be common to \( (A_{\mu+1}, B_{\mu+1}) \) by (21) and hence to all the further pairs \( (A, B) \).

**Theorem 12.** The convergents \( s_\nu(x) \) approach \( x \) with errors which are alternately positive and negative, but whose absolute values tend steadily to zero and are less than

\[ (x_\nu + 1) \tan \varphi, \]

where \( \varphi \) is the smaller of \( x^{1/2} \) and \( 3^{3/2} \).
Proof. By definition of \( \sigma \),
\[
\arccot x = \arccot \sigma + (-1)^n \{ \arccot n_r - \arccot n_{r+1} + \ldots \}.
\]
Since \( \arccot u \) is a decreasing function of \( u \), we have
\[
(-1)^{r+1}(x - \sigma) \geq 0,
\]
which implies the first statement of the theorem. Moreover, by (27) and (4),
\[
| \arccot x - \arccot \sigma | \leq \arccot n_r < n_r^{-1} < n_r^{-2} < \ldots < n_r^{-2^{r-1}} < n_r^{-2^{r-2}}.
\]
Hence if \( n_0 = [x] > 1 \), we may write
\[
| \arccot x - \arccot \sigma | < 2^{-2^{r-2}}.
\]
If \( n_0 = [x] \leq 1 \), then, by (4), \( n_1 \geq 1, n_2 \geq 3 \). Therefore in this case
\[
| \arccot x - \arccot \sigma | < 3^{-2^{r-1}}.
\]
Hence in either case
\[
| \arccot x - \arccot \sigma | < 2^{-r},
\]
and the final statement of the theorem follows by taking the tangent of both sides of this inequality. It remains to show that the absolute value of the error tends steadily to zero. Denoting this absolute value by \( \Delta_r \), we have by (28) and (20)
\[
\Delta_r = | x - \sigma | = (-1)^{r+1}(x - \sigma) = \frac{x^2 + 1}{x - (-1)^{r}x}.
\]
To show that \( \Delta_r \) is greater than \( \Delta_{r+1} \), it suffices to show that
\[
(1 + x^2)(\Delta_{r+1}^{-1} - \Delta_r^{-1}) = x_{r+1} - x_r + (-1)^r 2x
\]
is positive. From (15) and (4)
\[
x_{r+1} > n_r x_r + 1, \quad n_r \geq 3 \quad (r \geq 2).
\]
Hence \( x_{r+1} - x_r > x_r - x_{r-1} \). It follows from (30) that
\[
(1 + x^2)(\Delta_{r+1}^{-1} - \Delta_r^{-1}) > x_{r+1} - x_r - 2x.
\]
To show that the right member is positive we separate two cases. If \( x > 1 \), then \( n_1 \geq 3, x_2 > 3x_1 + 1, x_1 > n_0 x + 1 \geq x + 1 \). Hence in this case
\[
x_2 - x_1 - 2x > 2(x + 1) + 1 - 2x = 3 > 0.
\]
If \( x < 1 \), let \( x^{-1} = \delta > 1 \). Then \( x_1 = \delta, n_1 = \delta - \epsilon \) (0 < \( \epsilon < 1 \)), \( x_2 = \epsilon^{-1}[\delta(\delta - \epsilon) + 1] \). Therefore
\[
x_2 - x_1 - 2x = (\delta^2 + 1)(\delta - 2\epsilon)/\delta \epsilon.
\]
If \( \delta > 2 \), this is positive. If \( 1 < \delta < 2 \) so that \( n_1 = \delta - \epsilon = 1 \), we have
\[
x_2 - x_1 - 2x = (\delta^2 + 1)(1 - \epsilon)/\delta \epsilon > 0.
\]
This completes the proof of the theorem.
The expression of a regular continued cotangent as an irregular continued fraction. The partial cotangents \( n_r \) of a number \( x \) may be used to represent \( x \) by an irregular continued fraction of special type as the following theorem shows.

**Theorem 13.** If \( n_0, n_1, \ldots \) are the partial cotangents generated by a real positive number \( x \), then

\[
x = n_0 + \frac{n_0^2 + 1}{n_1 - n_0} + \frac{n_1^2 + 1}{n_2 - n_1} + \frac{n_2^2 + 1}{n_3 - n_2} + \cdots.
\]

**Proof.** Let \( \epsilon \) be the fractional part of \( x \), so that \( x = n_r + \epsilon \). Substituting for \( x \) and solving for \( \epsilon \), we obtain

\[
\epsilon = \frac{n_r^2 + 1}{n_{r+1} - n_r + \epsilon_{r+1}}.
\]

Setting \( \nu = 0, 1, 2, \ldots \) in succession, we see that (31) follows from \( x = n_0 + \epsilon \).

It is clear also that the numbers \( A_{\nu+1} \) and \( B_{\nu+1} \) are the numerator and denominator of the \( \nu \)-th convergent of (31).

**Regular continued fraction for \( \xi \).** The number \( \xi \) defined by (8) may be expressed as a regular continued fraction as follows. Let \( n_0, n_1, n_2, \ldots \) be the partial cotangents of \( \xi \) so that

\[
n_{r+1} = n_r - n_r^2 + 1.
\]

We define integers \( a_r \) by

\[
a_0 = 1, \quad a_1 = 1, \quad a_2 = 2, \quad a_3 = 5, \quad a_4 = 34, \quad a_5 = 985
\]

and in general

\[
a_{r+1} = (n_r + n_{r-1} + 1)a_{r-1} \quad (r \geq 1),
\]

so that

\[
a_{r+1} = (n_r + n_{r-1} + 1)(n_{r-2} + n_{r-3} + 1)(n_{r-4} + n_{r-5} + 1) \cdots.
\]

Where the last factor is \( n_1 + n_0 + 1 = 2 \) or \( n_2 + n_1 + 1 = 5 \) according as \( \nu \) is even or odd. Then it is true that

\[
a_{r+1}a_r = n_{r+1} - n_r = n_r^2 + 1.
\]

This fact is true for \( \nu = 0 \), since \( n_0^2 + 1 = 1 \), while \( a_0a_1 = 1 \). If it is true for \( \nu = k - 1 \), it may be shown true for \( \nu = k \) as follows:

\[
a_{k+1}a_k = \frac{a_{k+1}}{a_{k-1}}a_{k-1}a_{k-2} = (n_k + n_{k-1} + 1)(n_k - n_{k-1})\]

\[
= n_k^2 + n_k - n_{k-1} - n_{k-1} = n_k^2 + 1.
\]
This establishes (35). Returning to (31) and using (32) and (35), we obtain

\[
\xi = 0 + \frac{a_0 a_1}{|a_0 a_1|} + \frac{a_1 a_2}{|a_1 a_2|} + \frac{a_2 a_3}{|a_2 a_3|} + \ldots \\
= \frac{1}{a_0} + \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \ldots \\
= \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{5} + \frac{1}{34} + \frac{1}{985} + \ldots \\
+ \frac{1}{115138} + \frac{1}{1116929202845} + \ldots.
\]

The successive convergents \(C_r/D_r\) to \(\xi\) are

\[
\frac{1}{1}, \quad \frac{1}{1}, \quad \frac{1}{2}, \quad \frac{16}{5}, \quad \frac{547}{34}, \quad \frac{538811}{34}, \quad \frac{620245817465}{985}, \quad \ldots.
\]

In decimals we have

\[
.59263 \quad 27182 \quad 01636 \quad 19710 \quad 40786 \quad 04995 \quad 70146 \quad 90842 \\
75407 \quad 19716 \quad 10710 \quad 99562 \quad 60815 \quad 82473 \quad 51869 \quad 72201 \quad \ldots.
\]

An investigation into the nature of the number \(\xi\). The writer has been unable to discover any simple connection between \(\xi\) and other known constants. As to the nature of \(\xi\), it is neither rational nor the root of a quadratic equation with rational coefficients, since its continued fraction is neither finite nor periodic. In what follows we show that \(\xi\) is not a root of a cubic equation with rational coefficients. We begin with

**Theorem 14.** Let \(a_r\) and \(D_r\) be the \(r\)-th partial quotient and the denominator of the \(r\)-th convergent of the continued fraction (36). Then \(a_r > D_r\) for \(r \leq 4\).

**Remark.** For \(r = 1, 2, 3\), we have \(a_r = D_r\).

**Proof.** The theorem is true for \(r = 4\) since \(a_4 = 34\), and \(D_4 = 27\). If the theorem is true for \(4 \leq r < k\), we may prove it true for \(r = k\) by showing that \(a_{k-2}^{-1}(a_k - D_k)\) is positive. Using the fundamental recursion formula

\[
D_{r+1} = a_r D_r + D_{r-1}
\]

for \(r = k - 1, k - 2, k - 3\) we have

\[
a_{k-2}^{-1}(a_k - D_k) = a_{k-2}^{-1}\left\{a_k - (a_{k-1} a_{k-2} + 1) D_{k-1} - a_{k-1} \frac{D_{k-2} - D_{k-1}}{a_{k-2}}\right\}.
\]

If in (38) we introduce the hypothesis of the induction \(D_{k-2} \leq a_{k-2}\), we get

\[
a_{k-2}^{-1}(a_k - D_k) \geq a_k a_{k-1}^{-1} - (a_{k-1} a_{k-2} + 1) - a_{k-1} a_{k-2}^{-1}.
\]
COTANGENT ANALOGUE OF CONTINUED FRACTIONS

By (33) and (35) we may write (39) in the form
\[ a_{k-2}^{-1}(a_k - D_k) \geq n_{k-1} + n_{k-2} + 1 - (n_{k-1} - n_{k-2} + 1) - (n_{k-2} + n_{k-3} + 1) \]
\[ = n_{k-2} - n_{k-3} - 1 = n_{k-3}^2 > 0. \]

Hence the induction is complete.

**Theorem 15.** The number \( \xi \) does not satisfy a cubic equation with rational coefficients.

*Proof.* By Theorem 14 and the familiar inequality
\[ \left| \frac{\xi - C_1}{D_1} \right| < \frac{1}{D_1 D_2}, \]
\[ \frac{1}{D_1 (D_1 a_1 + D_1 - 1)} < \frac{1}{D_1^2}, \]
it follows that the Diophantine inequality
\[ \left| \frac{\xi - x}{y} \right| < \frac{1}{y^3} \]
has infinitely many solutions in integers \((x, y)\). Now if \( \xi \) satisfied a cubic equation with rational coefficients, the cubic would be irreducible, since \( \xi \) is neither rational nor a root of a quadratic equation. By a theorem of Siegel\(^*\) the inequality (40) would in this case have only a finite number of solutions in integers \((x, y)\), contrary to fact.

To show that this type of argument cannot be used further to prove that \( \xi \) is not an algebraic number of degree \( > 3 \), we give

**Theorem 16.** If \( \varepsilon > 0 \) and if \( c \) is a positive constant, no matter how large, the Diophantine inequality
\[ \left| \frac{\xi - x}{y} \right| < \frac{c}{y^{3+\varepsilon}} \]
has only a finite number of solutions \((x, y)\).

We first prove two other theorems.

**Theorem 17.** For every \( k \) the sequence
\[ \frac{D_k}{a_k}, \frac{D_{k+2}}{a_{k+2}}, \frac{D_{k+4}}{a_{k+4}}, \ldots \]
tends to a limit.

*Proof.* Since
\[ \frac{D_{k+2N}}{a_{k+2N}} = \frac{D_k}{a_k} - \sum_{\ell=0}^{N-1} \left( \frac{D_{k+2\ell}}{a_{k+2\ell}} - \frac{D_{k+2(\ell+1)}}{a_{k+2(\ell+1)}} \right), \]

it is sufficient to show that this series tends to a limit as \( N \to \infty \). To examine its general term we replace \( k + 2\lambda \) by \( \nu \) for simplicity. Now

\[
\frac{D_{r+2}}{a_{r+2}} = \frac{D_{r+1}a_{r+1} + D_r}{a_{r+2}} = \frac{D_r a_r a_{r+1} + D_{r-1}a_{r+1} + D_r}{a_{r+2}} = \frac{D_r(n_r^2 + 2)}{a_{r+2}} + \frac{D_{r-1}a_{r+1}}{a_{r+2}}
\]

\[
= \frac{D_r}{a_r} \frac{a_r}{a_{r+2}} (n_r^2 + 2) + \frac{D_{r-1}}{a_{r+2}} (n_{r+1}^2 + 1).
\]

Hence the general term of the above series may be written

\[
\frac{D_r}{a_r} - \frac{D_{r+2}}{a_{r+2}} = \frac{D_r}{a_r} \left( 1 - \frac{a_r}{a_{r+2}} (n_r^2 + 2) \right) - \frac{D_{r-1}}{a_{r+2}} (n_{r+1}^2 + 1).
\]

By Theorem 14 it is sufficient to show that as \( \nu \) runs over all numbers of the same parity as \( k \) the two infinite series

\[
\sum \left( 1 - \frac{a_r}{a_{r+2}} (n_r^2 + 2) \right), \quad \sum \frac{a_{r-1}}{a_{r+2}} (n_{r+1}^2 + 1)
\]

converge. The first of these may be written

\[
\sum \left( 1 - \frac{n_{r+1} - n_r + 1}{n_{r+1} + n_r + 1} \right) = \sum \frac{2n_r}{n_{r+1} + n_r + 1} < 2 \sum \frac{1}{n_r},
\]

a rapidly convergent series. As for the second series we have

\[
\sum \frac{a_{r-1}}{a_r} \frac{n_{r+1}^2 + 1}{(n_{r+1} + n_r + 1)^2} < \sum \frac{a_{r-1}}{a_r} < \sum \frac{1}{a_r},
\]

which also converges with rapidity. This completes the proof.

The two sequences

\[
\frac{D_1}{a_1}, \frac{D_2}{a_2}, \frac{D_3}{a_3}, \ldots \quad \text{and} \quad \frac{D_2}{a_2}, \frac{D_3}{a_3}, \frac{D_4}{a_4}, \ldots
\]

tend to different limits. In fact we find

\[
\frac{D_3}{a_3} = .7898114735158, \quad \frac{D_5}{a_5} = .9370558376, \quad \frac{D_7}{a_7} = .9370280114.
\]

These two limits we denote by \( R_0 \) and \( R_1 \). That is,

\[
R_0 = \lim_{\nu \to \infty} \frac{D_2}{a_2}, \quad \text{and} \quad R_1 = \lim_{\nu \to \infty} \frac{D_2}{a_2}.
\]

\[
R_0 = .78981147, \quad \text{and} \quad R_1 = .93702801.
\]
a limit as \( N \to \infty \). To examine
the two sequences above are both
strictly decreasing except for the fact that \( \frac{D_1}{a_1} = \frac{D_1}{a_2} \). We are now in a position
to prove

**Theorem 18.** If \( \nu \to \infty \), then

\[
\frac{D_1}{a_1} \left| \frac{\xi - C_{2\nu}}{D_{2\nu}} \right| \to R_0 \quad \text{and} \quad \frac{D_1}{a_1} \left| \frac{\xi - C_{2\nu+1}}{D_{2\nu+1}} \right| \to R_1.
\]

**Proof.** Since

\[
\xi = \lim_{\nu \to \infty} \frac{C_\nu}{D_\nu},
\]

we may write

\[
\xi = \frac{C_0}{D_0} + \frac{C_{\nu+1}}{D_{\nu+1}} - \frac{C_1}{D_1} + \frac{C_{\nu+2}}{D_{\nu+2}} - \frac{C_{\nu+1}}{D_{\nu+1}} + \cdots
\]

Using the fundamental relation

\[
C_\nu D_{\nu-1} - C_{\nu-1} D_\nu = (-1)^{\nu-1},
\]

we obtain

\[
\left| \frac{\xi - C_{\nu}}{D_{\nu}} \right| = \frac{D_{\nu+1}^2}{D_{\nu+1} D_{\nu+2}} \frac{D_{\nu+1}^2}{D_{\nu+1} D_{\nu+2}} + \frac{D_{\nu+1}^2}{D_{\nu+1} D_{\nu+2}} \frac{D_{\nu+1}^2}{D_{\nu+1} D_{\nu+2}} - \cdots
\]

The first term on the right may be shown to tend to \( R_0 \) or \( R_1 \) as follows:

\[
\frac{D_{\nu+1}^2}{D_{\nu+1}^2} = \frac{D_{\nu+1} a_\nu + D_{\nu+1} a_{\nu+1} a_\nu}{D_{\nu+1}^2} = \frac{a_\nu}{a_{\nu+1}} + \frac{D_{\nu+1}}{D_{\nu+2}}.
\]

As \( \nu \) tends to infinity through integers of the same parity, \( D_{\nu+1}^2 \) tends rapidly
to zero, while \( a_\nu / D_\nu \) tends to \( R_0^\nu \) or \( R_1^\nu \) according as the value of \( \nu \) is even or odd
by Theorem 17.

Each of the other terms of (42) tends to zero as \( \nu \to \infty \) since for \( \lambda > 0 \)

\[
\frac{D_{\nu+1}^2}{D_{\nu+1} D_{\nu+2}} \leq \frac{D_{\nu+1}^2}{D_{\nu+1} D_{\nu+2}} < \frac{D_{\nu+1}^2}{D_{\nu+1} D_{\nu+2}} < \frac{D_{\nu+1}^2}{D_{\nu+1} D_{\nu+2}} < \frac{D_{\nu+1}}{D_{\nu+1} D_{\nu+2}} < \frac{1}{a_\nu} < \frac{1}{a_{\nu+1}}.
\]

by Theorem 14. Hence the theorem is proved.

Theorem 16 now follows from Theorem 18 and from the fact that the convergents \( C_\nu / D_\nu \) are the fractions of best approximation to \( \xi \).

We conclude with a theorem concerning the above-mentioned limits \( R_0 \)
and \( R_1 \).

**Theorem 19.**

\[
R_0 R_1 = \frac{1}{1 + \xi^2}.
\]

For the proof of this theorem we need the following result of interest in itself.

**Theorem 20.** If \( C_\nu / D_\nu \) is the \( \nu \)-th convergent of (36), then

\[
C_\nu C_{\nu+1} + D_\nu D_{\nu+1} = n_{\nu+1}.
\]
Proof. Let $A_{r+1}$ and $B_{r+1}$ be the numerator and denominator of the $(r+1)_{st}$ convergent $c_{r+1}$ of the continued cotangent (8) defining $\xi$, or, what is the same, the numerator and denominator of the $r$-th convergent of the continued fraction (31). From the theory of irregular continued fractions we have in view of (32) the following recursion formulas for the $A$'s and the $B$'s:

\begin{align}
A_{r+1} &= (n_{r-1}^2 + 1)(A_r + A_{r-1}), \quad (43)
\end{align}

\begin{align}
B_{r+1} &= (n_{r-1}^2 + 1)(B_r + B_{r-1}). \quad (44)
\end{align}

We now prove that

\begin{align}
A_{r+1} &= C_0a_0a_{r-1}a_{r-3} \cdots a_0, \quad (45)
\end{align}

\begin{align}
B_{r+1} &= D_0a_0a_{r-1}a_{r-3} \cdots a_0. \quad (46)
\end{align}

In fact, (45) is true for $r = 0$, since $A_1 = n_0 = 0$ and $C_0 = 0$. If (45) is true for $r < \mu$ we may prove it for $r = \mu$ as follows. By (43) and the hypothesis of induction

\begin{align}
A_{\mu+1} &= (n_{\mu-1}^2 + 1)(C_{\mu-1}a_{\mu-1}a_{\mu-3} \cdots a_0 + C_{\mu-2}a_{\mu-2} \cdots a_0)
\end{align}

\begin{align}
&= (n_{\mu-1}^2 + 1)a_{\mu-2}a_{\mu-3} \cdots a_0(C_{\mu-1}a_{\mu-1} + C_{\mu-2}) = a_\mu a_{\mu-1}a_{\mu-3} \cdots a_0 C_\mu
\end{align}

by (35). (46) is established in the same way. By Theorem 11

\begin{align}
A_{r+1}A_{r+2} + B_{r+1}B_{r+2} = n_{r+1}(n_r^2 + 1)(n_{r-1}^2 + 1) \cdots (n_0^2 + 1), \quad (47)
\end{align}

while by (45) and (46)

\begin{align}
A_{r+1}A_{r+2} + B_{r+1}B_{r+2} = (C_rC_{r+1} + D_rD_{r+1})(a_{r+1}a_r)(a_0a_{r-1}) \cdots (a_0a_0)a_0
\end{align}

\begin{align}
&= (C_rC_{r+1} + D_rD_{r+1})(n_{r}^2 + 1)(n_{r-1}^2 + 1) \cdots (n_0^2 + 1)a_0.
\end{align}

If we compare this with (47), the theorem is seen to follow from $a_0 = 1$. In the same way Theorem 10 yields

Theorem 21.

\begin{align}
C_r^2 + D_r^2 = a_{r+1}.
\end{align}

Theorem 19 is now a simple consequence of Theorem 20. In fact, we have by definition of $R_0$ and $R_1$

\begin{align}
R_0R_i(l_r^2 + 1) &= \lim_{r \to \infty} \left( \frac{D_r}{a_r} \frac{D_{r+1}}{a_{r+1}} \left( \frac{C_r}{D_r} C_{r+1} + 1 \right) \right),
\end{align}

\begin{align}
&= \lim_{r \to \infty} \left( \frac{C_r}{D_r} C_{r+1} + 1 \right) = \lim_{r \to \infty} \frac{n_{r+1}}{n_{r+1} - n_r} = 1.
\end{align}

Regular continued cotangents of familiar constants. The converse problem of discovering a law enjoyed by the partial cotangents of the regular continued
cotangent analogue of continued fractions

Cotangent expansion of a familiar constant appears to be even more difficult than in continued fractions. There are no periodic regular continued cotangents in view of (4). In fact, a periodic continued cotangent would not converge. Hence equation (22) cannot be used as in continued fractions to study the roots of a quadratic equation with rational coefficients. Furthermore, it is practically impossible to find more than 6 or 8 partial cotangents of a given irrational number. By Theorem 12, ten terms of the continued cotangent expansion of a number \( x \) between 10 and 11 would give \( x \) correctly to more than 1000 decimal places, 20 terms would give more than a million digits. This dependence of the continued cotangent expansion upon the "size" of \( x \) is brought out more sharply by the fact that two numbers \( x_1 \) and \( x_2 \) which merely differ by an integer may have widely different continued cotangent expansions while their continued fraction expansions are essentially the same. Thus, for example, 13/25 = cot (arc cot 0 - arc cot 1 + arc cot 3 - arc cot 44), while 5 + (13/25) = cot (arc cot 5 - arc cot 55).

The writer has been unable to discover any combination of familiar constants whose regular continued cotangent expansion is in any way predictable; that is, we have found nothing comparable with

\[
\frac{3 - e}{e - 1} = \frac{1}{6} + \frac{1}{10} + \frac{1}{14} + \cdots + \frac{1}{4n + 2} + \cdots
\]

or with the irregular continued cotangent

\[
2 + \sqrt{2} = \cot (\text{arc cot } 3 + \text{arc cot } 17 + \text{arc cot } 99 + \text{arc cot } 577 + \cdots)
\]

whose partial cotangents satisfy the difference equation \( n_{r+1} = 6n_r - n_{r-1} \).

The continued cotangents for \( \sqrt{2}, \pi \) and \( \varepsilon \) begin as follows:

\[
\sqrt{2} = \cot (\text{arc cot } \frac{1}{2} - \text{arc cot } 5 + \text{arc cot } 36 - \text{arc cot } 3406 \\
+ \text{arc cot } 14694817 - \text{arc cot } 727050997716715 + \cdots),
\]

\[
\pi = \cot (\text{arc cot } 3 - \text{arc cot } 73 + \text{arc cot } 8599 - \text{arc cot } 400091364 + \cdots),
\]

\[
\varepsilon = \cot (\text{arc cot } 2 - \text{arc cot } 8 + \text{arc cot } 75 - \text{arc cot } 8949 \\
+ \text{arc cot } 11964723 \cdots).
\]

Although this paper is concerned with developing the general properties of regular continued cotangents, the reader cannot have failed to notice that many of the theorems have number-theoretic implications. The applications of the above theory to Diophantine analysis will be given in another paper.

An interesting generalization of the regular continued cotangent is an expansion of the form

\[
\cot \sum_{s=0}^{\infty} e_s \text{arc cot } n_s,
\]
in which \( c_i \) are \( \pm 1 \) and the \( p_i \) satisfy certain inequalities. This is called a "semi-regular" continued cotangent and has many properties in common with the semi-regular continued fraction

\[
g_0 \pm \frac{1}{|q_1|} \pm \frac{1}{|q_2|} \pm \ldots
\]

A discussion of semi-regular continued cotangents will appear later. However, if the coefficients \( c_i \) of (48) are unrestricted, the analogy with continued fractions breaks down.

Lehigh University.