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We have thus $(2\theta^2+1)^2=\theta^2$, that is, $4\theta^4+3\theta^2+1=0$ or $4k^2+3k+1=0$, or else $(\theta^2+2)^2=\theta^3$, that is, $\theta^4+3\theta^2+4=0$ or $k^2+3k+4=0$; viz. the equation in k is

$$(4k^2+3k+1)(k^2+3k+4)=0,$$

these being in fact the values of k given by the modular equation on putting therein $\Omega=1$.

The equation of the order 32 thus contains the factor $\{(\Omega,\ 1)^4\}$ at least twice, and it does not contain either the factor $\Omega-1$, or the factor $\{(\Omega,\ 1)^6\}$ belonging to the quintic transformation; it may be conjectured that the factor $\{(\Omega,\ 1)^4\}$ presents itself six times, and that the form is

$$\{(\Omega,\ 1)^4\}^6(\Omega,\ 1)^8=0;$$

but I am not able to verify this, and I do not pursue the discussion further.

22. The foregoing considerations show the grounds of the difficulty of the purely algebraical solution of the problem; the required results, for instance the modular equation, are obtained not in the simple form, but accompanied with special factors of high order. The transcendental theory affords the means of obtaining the results in the proper form without special factors; and I proceed to develop the theory, reproducing the known results as to the modular and multiplier equations, and extending it to the determination of the transformation-coefficients α, β, ...

The Modular Equation. Art. Nos. 23 to 28.

23. Writing, as usual, $q=e^{-\frac{\pi K'}{K}}$, we have u, a given function of q, viz.

$$\begin{aligned}u&=\sqrt{2}q^{\frac{1}{8}}\frac{1+q^2.1+q^4.1+q^6..}{1+q.1+q^3.1+q^5..}\\&=\sqrt{2}q^{\frac{1}{8}}(1-q+2q^2-3q^3+4q^4-6q^5+9q^6-12q^7+\ldots)\\&=\sqrt{2}q^{\frac{1}{8}}f(q)\text{ suppose};\end{aligned}$$

and this being so, the several values of v and of the other quantities in question are all given in terms of q.

The case chiefly considered is that of n an odd prime; and unless the contrary is stated it is assumed that this is so. We have then $n+1$ transformations corresponding to the same number $n+1$ of values of v; these may be distinguished as $v_0, v_1, v_2, \ldots, v_n$; viz. writing α to denote an imaginary nth root of unity, we have

$$v_0=(-)^{\frac{n^2-1}{8}}\sqrt{2}q^{\frac{n}{8}}f(q^n),\quad v_1=\sqrt{2}(\alpha q^{\frac{1}{n}})^{\frac{1}{8}}f(\alpha q^{\frac{1}{n}}),\quad v_2=\sqrt{2}(\alpha^2q^{\frac{1}{n}})^{\frac{1}{8}}f(\alpha^2q^{\frac{1}{n}}),\ \&c.,$$

$$v_n=\sqrt{2}q^{\frac{1}{8n}}f(q^{\frac{1}{n}}).$$

(Observe $(-)^{\frac{n^2-1}{8}}=+$ for $n=8p\pm1$, $-$ for $n=8p\pm3$.)

The occurrence of the fractional exponent $\frac{1}{8}$ is, as will appear, a circumstance of great importance; and it will be convenient to introduce the term "octicity," viz. an expression of the form $q^{\frac{f}{8}}F(q)$ ($f=0$, or a positive integer not exceeding 7, $F(q)$ a rational function of q) may be said to be of the octicity f.

24. The modular equation is of course

$$(v-v_0)(v-v_1)\ldots.(v-v_n)=0;$$

say this is

$$v^{n+1}-Av^n+Bv^{n-1}-\ldots=0,$$

so that $A=\Sigma v_0$, $B=\Sigma v_0v_1$, &c. In the development of these expressions, the terms having a fractional exponent, with denominator n, would disappear of themselves, as involving symmetrically the several nth roots of unity; and each coefficient would be of the form $q^{\frac{g}{8}}F(q)$, F a rational and integral function of q. It is moreover easy to see that, for the several coefficients A, B, C,......, g will denote the positive residue (mod. 8) of n, $2n$, $3n$,... respectively.

Hence assuming, as the fact is, that these coefficients are severally rational and integral functions of q, it follows that the form is

$$au^g+bu^{g+8}+cu^{g+16}+\ldots.,$$

g having the foregoing values for the several coefficients respectively. And it being known that the modular equation is as regards u of the order $=n+1$, there is a known limit to the number of terms in the several coefficients respectively. We have thus for each coefficient an identity of the form

$$A=au^g+bu^{g+8}+\ldots.,$$

where A and u being each of them given in terms of q, the values of the numerical coefficients a, b,.. can be determined; and we thus arrive at the modular equation.

25. It is in effect in this manner that the modular equations are calculated in Sohnke's Memoir. Various relations of symmetry in regard to (u, v) and other known properties of the modular equation are made use of in order to reduce the number of the unknown coefficients to a minimum; and (what in practice is obviously an important simplification) instead of the coefficients Σv_0, Σv_0v_1, &c., it is the sums of powers Σv_0, Σv_0^2, &c., which are compared with their expressions in terms of u, in order to the determination of the unknown numerical coefficients a, b,... The process is a laborious one (although less so than perhaps might beforehand have been imagined), involving very high numbers; it requires the development up to high powers of q, of the high powers of the before-mentioned function $f(q)$; and Sohnke gives a valuable Table, which I reproduce here; adding to it the three columns which relate to ϕq.

dun please enter all

1934 1935 1936 1937 1938 1939 1940 1941 1942 1943 1944

ind. of q.	$\phi q =$	$\phi^2 q =$	$\phi^{-2} q =$	$fq =$	$f^2 q =$	$f^3 q =$	$f^4 q =$	$f^5 q =$	$f^6 q =$	$f^7 q =$	$f^8 q =$	$f^9 q =$	$f^{10} q =$	$f^{11} q =$
0	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	+2	+4	−4	−1	−2	−3	−4	−5	−6	−7	−8	−9	−10	−11
2	0	+4	+12	+2	+5	+9	+14	+20	+27	+35	+44	+54	+65	+77
3	0	0	−32	−3	−10	−22	−40	−65	−98	−140	−192	−255	−330	−418
4	+2	+4	+76	+4	+18	+48	+101	+185	+309	+483	+718	+1026	+1420	+1914
5	0	+8	−168	−6	−32	−99	−236	−481	−882	−1498	−2400	−3672	−5412	−7733
6	0	0	+352	+9	+55	+194	+518	+1165	+2330	+4277	+7352	+11997	+18765	+28336
7	0	0	−704	−12	−90	−363	−1080	−2665	−5784	−11425	−20992	−36414	−60270	−95931
8	0	+4		+16	+144	+657	+2162	+5820	+13644	+28889	+56549	+103977	+181645	+304062
9	+2	+4		−22	−226	−1155	−4180	−12220	−30826	−69734	−145008	−281911	−518660	−911240
10	0	+8		+29	+346	+1977	+7840	+24802	+67107	+161735	+356388	+730953	+1413465	+2601786
11	0	0		−38	−522	−3312	−14328	−48880	−141444	−362271	−844032	−1822689	−3697960	−7120136
12	0	0		+50	+777	+5443	+25591	+93865	+289746	+786877	+1934534	+4390824	+9331565	+18766759
13	0	+8		−64	−1138	−8787	−44776	−176125	−578646	−1662927	−4306368	−10256508	−22800050	−47830486
14	0	0		+82	+1648	+13968	+76918	+323685	+1129527	+3428770	+9337704	+23303025	+54112825	+118270746
15	0	0		−105	−2362	−21894	−129952	−583798	−2159774	−6913760	−19771392	−51631227	−125090220	−284527793
16	+2	+4		+132	+3348	+33873	+216240	+1035060	+4052721	+13660346	+40965362	+111804966	+282298020	+667553898
17	0	+8		−166	−4704	−51795	−354864	−1806600	−7474806	−26492361	−83207976	−237074742	−623185010	−1530587256
18	0	+4		+208	+6554	+78345	+574958	+3108085	+15063859*	+50504755	+165944732	+493063403	+1348033540	+3435726536
19	0	0		−258	−9056	−117412								
20	0	+8		+320	+12425	+174033								
21	0	0		−395	−16932	−255945								
22	0	0		+484										
23	0	0		−592										
24	0	0		+722										
25	+2	+12		−876										
26	0	+8		+1060										

* [Wrongly given by Sohnke as +3108085.]

26	0 + 8		+1060

* [Wrongly given by Sohnke as +3108085.]

ind. of q.	$f^{12}q =$	$f^{13}q =$	$f^{14}q =$	$f^{15}q =$	$f^{16}q =$	$f^{17}q =$	$f^{18}q =$	$f^{19}q =$	$f^{20}q =$
0	1	1	1	1	1	1	1	1	1
1	− 12	− 13	− 14	− 15	− 16	− 17	− 18	− 19	− 20
2	+ 90	+ 104	+ 119	+ 135	+ 152	+ 170	+ 189	+ 209	+ 230
3	− 520	− 637	− 770	− 920	− 1088	− 1275	− 1482	− 1710	− 1960
4	+ 2523	+ 3263	+ 4151	+ 5205	+ 6444	+ 7888	+ 9558	+ 11476	+ 13665
5	− 10764	− 14651	− 19558	− 25668	− 33184	− 42330	− 53352	− 66519	− 82124
6	+ 41534	+ 59345	+ 82936	+ 113675	+ 153152	+ 203201	+ 265923	+ 343710	+ 439270
7	− 147720	− 221091	− 322828	− 461265	− 646528	− 890800	− 1208610	− 1617147	− 2136600
8	+ 490869	+ 768131	+ 1169847	+ 1739710	+ 2533070	+ 3619334	+ 5084478	+ 7034047	+ 9596460
9	− 1539472	− 2514551	− 3988292	− 6164345	− 9311664	− 13780540	− 20021534	− 28607673	− 40260300
10	+ 4592430	+ 7818200	+ 12896562	+ 20690964	+ 32387616	+ 49590581	+ 74438388	+ 109745767	+ 159174524
11	− 13111632	− 23233535	− 39809574	− 66222405	− 107299904	− 169812320	− 263104686		
12	+ 36006362	+ 66328964	+ 117921321	+ 203173760	+ 340436664	+ 556366922	+ 889020813		
13	− 95497116	− 182681916	− 336630840	− 600165795	− 1039026144	− 1752038020	− 2884990266		
14	+ 245457000	+ 487098378	+ 929461993	+ 1713196575	+ 3061896704	+ 5323089708	+ 9026077050		
15	− 613183064	− 1261118313	− 2489690882	− 4740491107	− 8739810688	− 15653783345	− 27314626158		
16	+ 1492474572	+ 3178449222	+ 6486711301	+ 12748926285	+ 24229115109	+ 44679433473	+ 80177033781*		
17	− 3546915228	− 7815313766	− 16475721276	− 33400680615	− 65390485328	− 124069449335	− 228831885054		
18	+ 8245677110	+ 18783535199	+ 40874694490	+ 85415669230	+ 172155210320	+ 335888162944	+ 636376573943		

* [In Sohnke, the figure 1 has] dropped out.