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2425 ✓

Mr Joffe

Sums of Like Powers of Natural Nos

QJM 46 33–51 1914

$$S_n(x) = 1^n + 2^n + \dots + x^n$$

as a p.s. in (i) x, in (ii) $2x+1$, in (iii) x^2+x

(i) $$S_n(x) = \frac{1}{n+1} x^{n+1} + \frac{1}{2} x^n + \frac{n}{2!} B_1 x^{n-1} - \frac{n(n-1)(n-2)}{4!} B_2 x^{n-2} + \dots$$

where B_i are Bernoulli nos

(ii) $$S_{2n-1}(x) = \frac{1}{2^{2n}.2n} \Big\{ (2x+1)^{2n} - (2n)_2 \beta_1 (2x+1)^{2n-2} + \dots$$
$$\dots + (-1)^{n-1} (2n)_{2n-2} \beta_{n-1} (2x+1)^2 + (-1)^n \alpha_n \Big\}$$

$S_{2n} = \dots$

where $\alpha_r = 2(2^{2r}-1) B_r$

$\beta_r = (2^{2r}-2) B_r$

(iii) Sims

where

$$T_{2n+1,r} = (-1)^r \Big\{ 1 - r_1 \frac{2n+1}{1} \beta_1 + r_2 \frac{(2n+1)(2n-1)}{1.3} \beta_2 \dots \Big\}$$

Table 1 gives the %'s a_i in the expansion

$$S_n(x) = a_1 x^{n+1} + a_2 x^{n} + a_3 x^{n-1} - a_4 x^{n-3} + a_5 x^{n-5} - \dots$$

$$\dots + (-1)^{i-1} a_i x^{n+5-2i} + \dots$$

Table II. gives the coefficients b_i in the expansion

$$S_n(x) = \frac{1}{2^{n+1}}\{b_1 s^{n+1} - b_2 s^{n-1} + b_3 s^{n-3} - \ldots + (-1)^{i-1} b_i s^{n+3-2i} + \ldots\},$$

s denoting $2x+1$.

Table III. gives the coefficients c_i, according as n is *odd* or *even*, in the respective expansions:

$$S_n(x) = c_1 w^{\frac{1}{2}(n+1)} - c_2 w^{\frac{1}{2}(n-1)} + c_3 w^{\frac{1}{2}(n-3)} - \ldots + (-1)^{i-1} c_i w^{\frac{1}{2}(n+3)-i} + \ldots,$$

$$S_n(x) = s\{c_1 w^{\frac{1}{2}n} - c_2 w^{\frac{1}{2}n-1} + c_3 w^{\frac{1}{2}n-2} - \ldots + (-1)^{i-1} c_i w^{\frac{1}{2}n-(i-1)} + \ldots\},$$

where $w = x^2 + x$.

§ 9. As preliminary figures we need for the calculation of Tables I. and III. the series of values of B_i, and for Table II. the series $(2^{2i}-2)B_i$ and $\frac{2^{2i}-1}{i}B_i$, the first of which, as we have seen, Dr. Glaisher denotes by β_i and the second corresponds to his $\frac{\alpha_i}{2i}$ and will be denoted here by α_i'. The calculations for the first 25 powers involve values of B_1 to B_{13} inclusive, and these values, together with the corresponding ones for β_i and α_i', are embodied in the three columns of the following Table.

i	B_i	β_i	α_i'
1	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$
2	$\frac{1}{30}$	$\frac{7}{15}$	$\frac{1}{4}$
3	$\frac{1}{42}$	$\frac{31}{21}$	$\frac{1}{2}$
4	$\frac{1}{30}$	$\frac{127}{15}$	$\frac{17}{8}$
5	$\frac{5}{66}$	$\frac{2555}{33}$	$\frac{31}{2}$
6	$\frac{691}{2730}$	$\frac{1414477}{1365}$	$\frac{691}{4}$

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i	B_i	β_i	α_i'
7	$\frac{7}{6}$	$\frac{57337}{3}$	$\frac{5461}{2}$
8	$\frac{3617}{510}$	$\frac{118518239}{255}$	$\frac{929569}{16}$
9	$\frac{43867}{798}$	$\frac{5749691557}{399}$	$\frac{3202291}{2}$
10	$\frac{174611}{330}$	$\frac{91546277357}{165}$	$\frac{221930581}{4}$
11	$\frac{854513}{138}$	$\frac{1792042792463}{69}$	$\frac{4722116521}{2}$
12	$\frac{236364091}{2730}$	$\frac{1982765468311237}{1365}$	$\frac{968383680827}{8}$
13	$\frac{8553103}{6}$	$\frac{286994504449393}{3}$	$\frac{14717667114151}{2}$

disagrees with my value: check! Probably OK

The correctness of the results in these three columns was tested by applying the formula: $B_i + \beta_i = i\alpha_i'$, *e.g.*, in the third line: $\frac{1}{42} + \frac{31}{21} = 3 \times \frac{1}{2}$.

Calculation of Table I., § 10.

§ 10. The first column of Table I. consists of the fractions $\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{26}$. The second column is uniformly $\frac{1}{2}$. The third column begins in the second row with $\frac{1}{6}$, *i.e.*, B_1, and each succeeding column begins two rows lower with B_2, B_3, etc., respectively. The successive numbers, beginning with the third, in each (i^{th}) row are obtained by multiplying the corresponding numbers in the next preceding ($i-1^{\text{th}}$) row by $\frac{i}{i-1}, \frac{i}{i-3}, \frac{i}{i-5}, \ldots$ respectively. For instance, the fractions in the ninth row, $\frac{3}{4}, \frac{7}{10}, \frac{1}{2}$ and $\frac{3}{20}$, are obtained from those in the eighth row, $\frac{2}{3}, \frac{7}{15}, \frac{2}{9}$ and $\frac{1}{30}$, by multiplying the latter by $\frac{9}{8}, \frac{9}{6}, \frac{9}{4}$ and $\frac{9}{2}$ respectively.

In order to avoid accumulation of errors, every fifth row was verified by utilizing the fact that the sum of the numbers in any row, taking them alternately positive and negative, equals 0; *e.g.*, in the tenth row we have

$$\frac{1}{11} - \frac{1}{2} + \frac{5}{6} - 1 + 1 - \frac{1}{2} + \frac{5}{66} = 0.$$

Calculation of Table II., § 11.

§ 11. The first column of Table II. is the same as the first column of Table I., *i.e.*, $\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{26}$. The first two numbers in the second column are α_1' and β_1; in the third column α_2' and β_2, and, in general, in the i^{th} column α'_{i-1} and β_{i-1}. The second column, excepting its first number, which is $\frac{1}{2}$, consists of the fractions $\frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \ldots$. The successive numbers, except the first two and last one, in each (i^{th}) row are obtained by multiplying the corresponding numbers in the next preceding ($i-1^{th}$) row by $\frac{i}{i-3}, \frac{i}{i-5}, \frac{i}{i-7}, \ldots$, respectively. For instance, the numbers in the tenth row, 14, 62, 127, are obtained from those in the ninth row, $\frac{49}{5}$, 31, $\frac{381}{10}$, by multiplying the latter by $\frac{10}{7}, \frac{10}{5}, \frac{10}{3}$ respectively.

Again, in order to avoid accumulation of errors, every fifth row was verified by the same test as in Table I. For instance, in the tenth row we find

$$\frac{1}{11} - \frac{5}{3} + 14 - 62 + 127 - \frac{2555}{33} = 0.$$

As an additional test, a simultaneous verification of Tables I. and II. was made by comparing their last rows with each other and observing the following relations:

$$\frac{25}{6} : \frac{25}{12} = 2 = 2^2 - 2,$$

$$\frac{805}{3} : \frac{115}{6} = 14 = 2^4 - 2,$$

$$\frac{39215}{3} : \frac{1265}{6} = 62 = 2^6 - 2, \text{ etc.,}$$

the last being.

$$\frac{9913827341556185}{546} : \frac{1181820455}{1092} = 16777214 = 2^{24} - 2.$$

Calculation of Table III., §§ 12–13.

§ 12. The first column of Table III. contains in the *odd* rows the fractions $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots$, and in the *even* rows, $\frac{1}{6}, \frac{1}{10}, \frac{1}{14}, \ldots$. Thus, the second number in the first column is $\frac{1}{6}$ or B_1; the second column begins in the fourth row with B_2, the third begins in the sixth row with B_3, etc. The successive numbers, beginning with the second, in each *odd* ($2i+1^{th}$) row are obtained by multiplying the corresponding numbers in the next preceding *even* ($2i^{th}$) row by $\frac{2i+1}{i}, \frac{2i+1}{i-1}, \frac{2i+1}{i-2}, \ldots$ respectively. For instance, the fractions in the eleventh row, $\frac{1}{3}, \frac{17}{24}, \frac{5}{6}, \frac{5}{12}$, are obtained from those in the tenth row, $\frac{5}{33}, \frac{17}{66}, \frac{5}{22}, \frac{5}{66}$, by multiplying the latter by $\frac{11}{5}, \frac{11}{4}, \frac{11}{3}, \frac{11}{2}$ respectively.

The *even* rows, however, cannot be obtained from the *odd* rows in as simple a manner as in the first case. The method employed for this purpose will best appear from the following illustration. The sixteenth row was derived from the fifteenth row, by arranging the calculations in six lines as follows:

1	$\frac{1}{16}$	$\frac{1}{2}$	$\frac{7}{3}$	$\frac{22}{3}$	$\frac{359}{24}$	$\frac{35}{2}$	$\frac{35}{4}$	
2	1	8	$\frac{112}{3}$	$\frac{352}{3}$	$\frac{718}{3}$	280	140	
3	0	$\frac{4}{17}$	$\frac{98}{51}$	$\frac{154}{17}$	$\frac{1465}{51}$	$\frac{3038}{51}$	$\frac{1237}{17}$	
4	1	$\frac{140}{17}$	$\frac{2002}{51}$	$\frac{6446}{51}$	$\frac{4557}{17}$	$\frac{17318}{51}$	$\frac{3617}{17}$	
5	34	30	26	22	18	14	10	
6	$\frac{1}{34}$	$\frac{14}{51}$	$\frac{77}{51}$	$\frac{293}{51}$	$\frac{1519}{102}$	$\frac{1237}{51}$	$\frac{3617}{170}$	$\frac{3617}{510}$

The first line contains the numbers of the fifteenth row. In the second line are entered the results of multiplying the first line by 16. The third line we begin with 0, and then copy the results, one by one, after having obtained them by multiplying the successive six numbers in the sixth line by $8\left(i.e., \frac{16}{2}\right)$, 7, 6, ... respectively. The fourth line is the sum of the second and third lines. The fifth contains the series of numbers beginning with 34 ($i.e.$, $2 \times 16 + 2$) and decreasing by 4. Finally, the sixth line is the quotient of the fourth line by the fifth. After the seventh number in the sixth line we write one more number which is $\frac{1}{3}$ of the seventh. The eight numbers thus obtained constitute the sixteenth row, and the correctness of calculations in this row is tested by the fact that the eighth number, $\frac{3617}{510}$, equals B_8, as entered originally.

§ 13. As a final test of the correctness of the whole Table III., the twenty-fifth row was verified independently. If we take $x = 1$, making $w = x^2 + x = 2$, then the formula

$$S_{2n-1}(x) = c_1 w^n - c_2 w^{n-1} + c_3 w^{n-2} - \ldots + (-1)^n c_{n-1} w^2$$

becomes $c_1 2^n - c_2 2^{n-1} + c_3 2^{n-2} - \ldots + (-1)^n c_{n-1} 2^2 = 1,$

or $\{\ldots[(2c_1 - c_2)\,2 + c_3]\,2 - c_4 \ldots + (-1)^n c_{n-1}\}\,2^2 = 1.$

The work of applying this formula to the twenty-fifth row may be arranged in three lines as follows:

$$\begin{array}{lcccccccc}
(1) & \frac{1}{26} & -\frac{11}{12} & \frac{83}{6} & -\frac{649}{4} & \frac{9185}{6} & \ldots & -\frac{1181820455}{1092}, \\[6pt]
(2) & 0 & \frac{1}{13} & -\frac{131}{78} & \frac{316}{13} & -\frac{7173}{26} & \ldots & \frac{295455182}{273}, \\[6pt]
(3) & \frac{1}{26} & -\frac{131}{156} & \frac{158}{13} & -\frac{7173}{52} & \frac{48943}{39} & \ldots & \frac{1}{4},
\end{array}$$

where line (1) is a copy of the twenty-fifth row, with signs alternately *plus* and *minus*; each number in line (2), except the first which is 0, is double the number in the preceding column in line (3), and line (3) is the sum of lines (1) and (2). The last number of the third line comes out $\frac{1}{4}$, as it should be.

TABLE I.

Coefficients a_i of $S_n(x)$ expressed as a power-series in x.

$$S_n(x) = a_1 x^{n+1} + a_2 x^n + a_3 x^{n-1} - a_4 x^{n-3} + \ldots + (-1)^{i-1} a_i x^{n+k-2i} + \ldots.$$

n	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
1	$\frac{1}{2}$	$\frac{1}{2}$						
2	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$					
3	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$					
4	$\frac{1}{5}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{30}$				
5	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{5}{12}$	$\frac{1}{12}$				
6	$\frac{1}{7}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{42}$			
7	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{7}{12}$	$\frac{7}{24}$	$\frac{1}{12}$			
8	$\frac{1}{9}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{7}{15}$	$\frac{2}{9}$	$\frac{1}{30}$		
9	$\frac{1}{10}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{7}{10}$	$\frac{1}{2}$	$\frac{3}{20}$		
10	$\frac{1}{11}$	$\frac{1}{2}$	$\frac{5}{6}$	1	1	$\frac{1}{2}$	$\frac{5}{66}$	
11	$\frac{1}{12}$	$\frac{1}{2}$	$\frac{11}{12}$	$\frac{11}{8}$	$\frac{11}{6}$	$\frac{11}{8}$	$\frac{5}{12}$	
12	$\frac{1}{13}$	$\frac{1}{2}$	1	$\frac{11}{6}$	$\frac{22}{7}$	$\frac{33}{10}$	$\frac{5}{3}$	$\frac{691}{2,730}$
13	$\frac{1}{14}$	$\frac{1}{2}$	$\frac{13}{12}$	$\frac{143}{60}$	$\frac{143}{28}$	$\frac{143}{20}$	$\frac{65}{12}$	$\frac{691}{420}$

TABLE I. (*Continued*).

Coefficients a_i of $S_n(x)$ expressed as a power-series in x.

$$S_n(x) = a_1x^{n+1} + a_2x^n + a_3x^{n-1} - a_4x^{n-3} + \dots + (-1)^{i-1}a_ix^{n+5-2i} + \dots$$

n	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
14	1/15	1/2	7/6	91/30	143/18	143/10	91/6	691/90	7/6
15	1/16	1/2	5/4	91/24	143/12	429/16	455/12	691/24	35/4
16	1/17	1/2	4/3	14/3	52/3	143/3	260/3	1,382/15	140/3
17	1/18	1/2	17/12	17/3	221/9	2,431/30	1,105/6	11,747/45	595/3
18	1/19	1/2	3/2	34/5	34	663/5	1,105/3	23,494/35	714
19	1/20	1/2	19/12	323/40	323/7	4,199/20	4,199/6	223,193/140	2,261
20	1/21	1/2	5/3	19/2	1,292/21	323	41,990/33	223,193/63	6,460
21	1/22	1/2	7/4	133/12	323/4	969/2	146,965/66	223,193/30	33,915/2
22	1/23	1/2	11/6	77/6	209/2	3,553/5	11,305/3	223,193/15	124,355/3
23	1/24	1/2	23/12	1,771/120	4,807/36	81,719/80	37,145/6	5,133,439/180	572,033/6
24	1/25	1/2	2	253/15	506/3	14,421/10	29,716/3	10,266,878/195	208,012
25	1/26	1/2	25/12	115/6	1,265/6	24,035/12	185,725/12	25,667,195/273	1,300,075/3

TABLE I. (*Continued*).

Coefficients a_i of $S_n(x)$ expressed as a power-series in x.

$$S_n(x) = a_1x^{n+1} + a_2x^n + a_3x^{n-1} - a_4x^{n-3} + \dots + (-1)^{i-1}a_ix^{n+5-2i} + \dots$$

n	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}
14					
15					
16	3,617/510				
17	3,617/60				
18	3,617/10	43,867/798			
19	68,723/40	43,867/84			
20	68,723/10	219,335/63	174,611/330		
21	481,061/20	219,335/12	1,222,277/220		
22	755,953/10	482,537/6	1,222,277/30	851,513/138	
23	17,386,919/80	11,098,351/36	28,112,371/120	854,513/12	
24	17,386,919/30	22,196,702/21	28,112,371/25	1,709,026/3	236,364,091/2,730
25	17,386,919/12	277,458,775/84	28,112,371/6	21,362,825/6	1,181,820,455/1,092

Table II.

Coefficients b_i of $S_n(x)$ expressed as a power-series in $s=2x+1$.

$$S_n(x)=\frac{1}{2^{n+1}}\{b_1 s^{n+1}-b_2 s^{n-1}+b_3 s^{n-3}-b_4 s^{n-5}+\dots+(-1)^{i-1}b_i s^{n+3-2i}+\dots\}.$$

n	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9
1	1/2	1/2							
2	1/3	1/3							
3	1/4	1/2	1/4						
4	1/5	2/3	7/15						
5	1/6	5/6	7/6	1/2					
6	1/7	1	7/3	31/21					
7	1/8	7/6	49/12	31/6	17/8				
8	1/9	4/3	98/15	124/9	127/15				
9	1/10	3/2	49/5	31	381/10	31/2			
10	1/11	5/3	14	62	127	2,555/33			
11	1/12	11/6	77/4	341/3	1,397/4	2,555/6	691/4		
12	1/13	2	77/3	1,364/7	4,191/5	5,110/3	1,414,477/1,365		
13	1/14	13/6	1,001/30	4,433/14	18,161/10	33,215/6	1,414,477/210	5,461/2	
14	1/15	7/3	637/15	4,433/9	18,161/5	46,501/3	1,414,477/45	57,337/3	
15	1/16	5/2	637/12	4,433/6	54,483/8	232,505/6	1,414,477/12	286,685/2	929,569/16

Table II. (*Continued.*)

Coefficients b_i of $S_n(x)$ expressed as a power-series in $s=2x+1$.

$$S_n(x)=\frac{1}{2^{n+1}}\{b_1 s^{n+1}-b_2 s^{n-1}+b_3 s^{n-3}-b_4 s^{n-5}+\dots+(-1)^{i-1}b_i s^{n+3-2i}+\dots\}.$$

n	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9
16	1/17	8/3	196/3	3,224/3	36,322/3	265,720/3	5,657,908/15	2,293,480/3	118,518,239/255
17	1/18	17/6	238/3	13,702/9	308,737/15	564,655/3	48,092,218/45	9,747,290/3	118,518,239/30
18	1/19	3	476/5	2,108	168,402/5	1,129,310/3	96,184,436/35	11,696,748	118,518,239/5
19	1/20	19/6	2,261/20	20,026/7	533,273/10	2,145,689/3	456,876,071/70	37,039,702	2,251,846,541/20
20	1/21	10/3	133	80,104/21	82,042	42,913,780/33	913,752,142/63	105,827,720	2,251,846,541/5
21	1/22	7/2	931/6	10,013/2	123,063	75,099,115/33	456,876,071/15	277,797,765	15,762,925,787/10
22	1/23	11/3	539/3	6,479	902,462/5	11,553,710/3	913,752,142/15	2,037,183,610/3	24,770,311,951/5
23	1/24	23/6	12,397/60	149,017/18	10,378,313/40	18,981,095/3	10,508,149,633/90	4,685,522,303/3	569,717,174,873/40
24	1/25	4	3,542/15	31,372/3	1,831,467/5	30,369,752/3	42,032,598,532/195	3,407,652,584	569,717,174,873/15
25	1/26	25/6	805/3	39,215/3	3,052,445/6	94,905,475/6	105,081,496,330/273	21,297,828,650/3	569,717,174,873/6

TABLE II. (*Continued.*)

Coefficients b_i of $S_n(x)$ expressed as a power-series in $s=2x+1$.

$$S_n(x)=\frac{1}{2^{n+1}}\{b_1s^{n+1}-b_2s^{n-1}+b_3s^{n-3}-b_4s^{n-5}+\ldots+(-1)^{i-1}b_is^{n+3-2i}+\ldots\}.$$

n	b_{10}	b_{11}	b_{12}	b_{13}	b_{14}
17	$\frac{3,202,291}{2}$				
18	$\frac{5,749,691,557}{399}$				
19	$\frac{5,749,691,557}{42}$	$\frac{221,930,581}{4}$			
20	$\frac{57,496,915,570}{63}$	$\frac{91,546,277,357}{165}$			
21	$\frac{28,748,457,785}{6}$	$\frac{640,823,941,499}{110}$	$\frac{4,722,116,521}{2}$		
22	$\frac{63,246,607,127}{3}$	$\frac{640,823,941,499}{15}$	$\frac{1,792,042,792,463}{69}$		
23	$\frac{1,454,671,963,921}{18}$	$\frac{14,738,950,654,477}{60}$	$\frac{1,792,042,792,463}{6}$	$\frac{968,383,680,827}{8}$	
24	$\frac{5,818,687,855,684}{21}$	$\frac{29,477,901,308,954}{25}$	$\frac{7,168,171,169,852}{3}$	$\frac{1,982,765,168,311,237}{1,365}$	
25	$\frac{36,366,799,098,025}{42}$	$\frac{14,738,950,654,477}{3}$	$\frac{44,801,069,811,575}{3}$	$\frac{9,913,827,311,556,185}{546}$	$\frac{14,717,667,114,151}{2}$

TABLE III.

Coefficients c_i of $S_n(x)$ as a power-series in $w=x^2+x$.

$$n=odd,\ S_n(x)=c_1w^{\frac{1}{2}(n+1)}-c_2w^{\frac{1}{2}(n-1)}+c_3w^{\frac{1}{2}(n-3)}-\ldots+(-1)^{i-1}c_iw^{\frac{1}{2}(n+3)-i}+\ldots$$

$$n=even,\ S_n(x)=s\{c_1w^{\frac{1}{2}n}-c_2w^{\frac{1}{2}n-1}+c_3w^{\frac{1}{2}n-2}-\ldots+(-1)^{i-1}c_iw^{\frac{1}{2}n-(i-1)}+\ldots\}.$$

n	c_1	c_2	c_3	c_4	c_5	c_6
1	$\frac{1}{2}$					
2	$\frac{1}{6}$					
3	$\frac{1}{4}$					
4	$\frac{1}{10}$	$\frac{1}{30}$				
5	$\frac{1}{6}$	$\frac{1}{12}$				
6	$\frac{1}{14}$	$\frac{1}{14}$	$\frac{1}{42}$			
7	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{12}$			
8	$\frac{1}{18}$	$\frac{1}{9}$	$\frac{1}{10}$	$\frac{1}{30}$		
9	$\frac{1}{10}$	$\frac{1}{4}$	$\frac{3}{10}$	$\frac{3}{20}$		
10	$\frac{1}{22}$	$\frac{5}{33}$	$\frac{17}{66}$	$\frac{5}{22}$	$\frac{5}{66}$	
11	$\frac{1}{12}$	$\frac{1}{3}$	$\frac{17}{24}$	$\frac{5}{6}$	$\frac{5}{12}$	
12	$\frac{1}{26}$	$\frac{5}{26}$	$\frac{41}{78}$	$\frac{236}{273}$	$\frac{691}{910}$	$\frac{691}{2,730}$
13	$\frac{1}{14}$	$\frac{5}{12}$	$\frac{41}{30}$	$\frac{59}{21}$	$\frac{691}{210}$	$\frac{691}{420}$

TABLE III. (*Continued.*)

Coefficients c_i of $S_n(x)$ as a power-series in $w = x^2 + x$.

$n = odd,\ S_n(x) = c_1 w^{\frac{1}{2}(n+1)} - c_2 w^{\frac{1}{2}(n-1)} + c_3 w^{\frac{1}{2}(n-3)} - \ldots + (-1)^{i-1} c_i w^{\frac{1}{2}(n+3)-i} + \ldots$

$n = even,\ S_n(x) = s\{c_1 w^{\frac{1}{2}n} - c_2 w^{\frac{1}{2}n-1} + c_3 w^{\frac{1}{2}n-2} - \ldots + (-1)^{i-1} c_i w^{\frac{1}{2}n-(i-1)} + \ldots\}.$

n	c_1	c_2	c_3	c_4	c_5	c_6	c_7
14	$\frac{1}{30}$	$\frac{7}{30}$	$\frac{14}{15}$	$\frac{22}{9}$	$\frac{359}{90}$	$\frac{7}{2}$	$\frac{7}{6}$
15	$\frac{1}{16}$	$\frac{1}{2}$	$\frac{7}{3}$	$\frac{22}{3}$	$\frac{359}{24}$	$\frac{35}{2}$	$\frac{35}{4}$
16	$\frac{1}{34}$	$\frac{14}{51}$	$\frac{77}{51}$	$\frac{293}{51}$	$\frac{1,519}{102}$	$\frac{1,237}{51}$	$\frac{3,617}{170}$
17	$\frac{1}{18}$	$\frac{7}{12}$	$\frac{11}{3}$	$\frac{293}{18}$	$\frac{1,519}{30}$	$\frac{1,237}{12}$	$\frac{3,617}{30}$
18	$\frac{1}{38}$	$\frac{6}{19}$	$\frac{217}{95}$	$\frac{1,129}{95}$	$\frac{8,487}{190}$	$\frac{6,583}{57}$	$\frac{750,167}{3,990}$
19	$\frac{1}{20}$	$\frac{2}{3}$	$\frac{217}{40}$	$\frac{1,129}{35}$	$\frac{2,829}{20}$	$\frac{6,583}{15}$	$\frac{750,167}{840}$
20	$\frac{1}{42}$	$\frac{5}{14}$	$\frac{23}{7}$	$\frac{470}{21}$	$\frac{689}{6}$	$\frac{28,399}{66}$	$\frac{1,540,967}{1,386}$
21	$\frac{1}{22}$	$\frac{3}{4}$	$\frac{23}{3}$	$\frac{235}{4}$	$\frac{689}{2}$	$\frac{198,793}{132}$	$\frac{1,540,967}{330}$
22	$\frac{1}{46}$	$\frac{55}{138}$	$\frac{209}{46}$	$\frac{902}{23}$	$\frac{60,511}{230}$	$\frac{928,151}{690}$	$\frac{1,737,577}{345}$
23	$\frac{1}{24}$	$\frac{5}{6}$	$\frac{209}{20}$	$\frac{902}{9}$	$\frac{60,511}{80}$	$\frac{132,593}{30}$	$\frac{1,737,577}{90}$
24	$\frac{1}{50}$	$\frac{11}{25}$	$\frac{913}{150}$	$\frac{649}{10}$	$\frac{5,511}{10}$	$\frac{276,208}{75}$	$\frac{6,114,166}{325}$
25	$\frac{1}{26}$	$\frac{11}{12}$	$\frac{83}{6}$	$\frac{649}{4}$	$\frac{9,185}{6}$	$\frac{34,526}{3}$	$\frac{6,114,166}{91}$

TABLE III. (*Continued.*)

Coefficients c_i of $S_n(x)$ as a power-series in $w = x^2 + x$.

$n = odd,\ S_n(x) = c_1 w^{\frac{1}{2}(n+1)} - c_2 w^{\frac{1}{2}(n-1)} + c_3 w^{\frac{1}{2}(n-3)} - \ldots + (-1)^{i-1} c_i w^{\frac{1}{2}(n+3)-i} + \ldots$

$n = even,\ S_n(x) = s\{c_1 w^{\frac{1}{2}n} - c_2 w^{\frac{1}{2}n-1} + c_3 w^{\frac{1}{2}n-2} - \ldots + (-1)^{i-1} c_i w^{\frac{1}{2}n-(i-1)} + \ldots\}.$

n	c_8	c_9	c_{10}	c_{11}	c_{12}
14					
15					
16	$\frac{3,617}{510}$				
17	$\frac{3,617}{60}$				
18	$\frac{43,867}{266}$	$\frac{43,867}{798}$			
19	$\frac{43,867}{42}$	$\frac{43,867}{84}$			
20	$\frac{1,254,146}{693}$	$\frac{174,611}{110}$	$\frac{174,611}{330}$		
21	$\frac{627,073}{66}$	$\frac{1,222,277}{110}$	$\frac{1,222,277}{220}$		
22	$\frac{299,264}{23}$	$\frac{4,871,093}{230}$	$\frac{854,513}{46}$	$\frac{854,513}{138}$	
23	$\frac{299,264}{5}$	$\frac{4,871,093}{40}$	$\frac{854,513}{6}$	$\frac{854,513}{12}$	
24	$\frac{22,888,038}{325}$	$\frac{4,730,237}{26}$	$\frac{26,947,575}{91}$	$\frac{236,364,091}{910}$	$\frac{236,364,091}{2,730}$
25	$\frac{3,814,673}{13}$	$\frac{23,651,185}{26}$	$\frac{673,689,375}{364}$	$\frac{1,181,820,455}{546}$	$\frac{1,181,820,455}{1,092}$