Scan

add to ~~[illegible]~~ 7

sequences (same as his

letter)

Possible new ~~report~~ ~~NT378~~ A1787 A272 A89946 A152818 A6043 A6044 A259334

BIφM

[*From* Biometrika, *Vol.* 46. *Parts* 3 *and* 4, *December* 1959.]

OVERFLOW AT A TRAFFIC LIGHT

FRANK A. HAIGHT
Institute of Transportation and Traffic Engineering,
University of California, Los Angeles

1. Lights in general

Suppose vehicles of uniform length arrive at an intersection controlled by a traffic light, with arrival times which constitute a homogeneous Poisson process, with parameter λ. The effect of acceleration will be ignored; once a vehicle is near the traffic light it travels with a uniform speed S unless stopped. When a queue of vehicles is stopped by the light, headway (distance separation between corresponding parts of adjacent vehicles) is constant, and when the cars move off the headway will be a larger constant, denoted by K.

The letter β will be used to indicate the length of the red phase, so that the expected number of arrivals during a red phase will be $\lambda\beta$ for a fixed cycle light and $\lambda E(\beta)$ for a variable cycle light. During the green phase, when there is a queue of cars waiting, they are discharged with time separation $K/S = T$. Once the queue present at the beginning of the green phase has been emptied, the Poisson arrivals continue through the intersection without delay for the remainder of the green phase. Instead of characterizing the green phase by its length, we will use instead an integer N, which is the largest multiple of T contained in its length:

$$\text{length green} = \alpha = NT + \theta T \quad (0 < \theta < 1).$$

Thus, with a sufficiently long queue, at most N vehicles can be discharged during a green phase. N is, however, by no means the maximum number of vehicles which may pass through the intersection during a green phase, for once the queue is dissipated, the Poisson stream may send cars through with time separation $< T$. In some circumstances N will be a constant, and in others a stochastic variable governed by vehicle actuation on the side street.

In speaking of input and output, we do not refer to input and output to the intersection, but only to the congestion at the intersection. Thus, when there is no queue, a car may come into the intersection and pass out of it, leaving both input and output zero. It is essential to specify under what circumstances input is possible. It might be assumed that S is as great as the speed of arriving vehicles, and so input would be zero as soon as the last queued vehicle began to move, provided *that* vehicle cleared the intersection in a single green phase. Such an assumption seems to conflict with our experience and certainly leads to an unwieldy formulation. Consequently we will assume continued input in every circumstance except zero queue length.

2. Fixed cycle lights

Suppose β, T and N are constants. We will have occasion to use the Poisson probability, which will be abbreviated

$$P(u;\Delta) = e^{-\Delta}\Delta^u/u! \quad (u = 0, 1, 2, \ldots),$$

and the Borel–Tanner probability [cf. Borel (1942), Tanner (1953), Haight & Breuer (1960)], which will be abbreviated

$$R(u;r,\rho) = A(u,r)\,e^{-\rho u}\rho^{u-r} \quad (u = r, r+1, \ldots),$$

where
$$A(u,r) = \frac{r}{(u-r)!}\,u^{u-r-1}.$$

This represents the probability that exactly u members of a queue will be served before the queue first becomes empty, beginning with r members and with traffic intensity ρ, assuming Poisson input and regular discharge. In our application, during the green phase, ρ always equals λT [approximately $\alpha\lambda/N$], and will be omitted, writing $R(u; r)$.

The purpose of this paper is to compute the probability of Z cars being in the queue at the beginning of a red phase, when there were X cars in the queue at the beginning of the preceding green phase. Z will be called the overflow into the red phase. If $X > N$, the overflow must be non-zero, and will depend only on the input during the green phase. Letting

$$f(z; x) = \Pr[Z = z \mid X = x],$$

we have
$$f(z; x) = P(z-x+N; \alpha\lambda) \quad (x > N). \tag{1}$$

When $x \leqslant N$, $z = 0$, the formula is also fairly simple, and can be written in terms of cumulative Borel–Tanner probabilities as follows:

$$f(0; x) = \sum_{j=x}^{N} R(j; x) \quad (x \leqslant N). \tag{2}$$

In the remaining cases ($z > 0$, $x \leqslant N$), consider, for fixed x, the event that the overflow would be z, and that no vehicles would arrive after the light turned red in the period that it would take this overflow to clear. The probability of this is $f(z; x)\,e^{-\rho z}$, and it may be regarded as the probability that the queue would first vanish after $N+z$ vehicles and that the overflow was z. The unconditional probability of the queue first clearing after $N+z$ vehicles is of course $R(N+z; x)$, and therefore the difference $R(N+z; x) - f(z; x)\,e^{-\rho z}$ is the sum from $j = 1$ to $j = z-1$ of the probability that the queue would first vanish after $N+z$ vehicles and that the overflow is j, that is

$$\sum_{j=1}^{z-1} f(j; x)\,R(z; j),$$

from which we obtain

$$f(z; x) = e^{\rho z}\Big[R(N+z; x) - \sum_{j=1}^{z-1} R(z; j) f(j; x)\Big] \quad (z > 0,\ x \leqslant N). \tag{3}$$

Equations (1), (2) and (3) taken together give a formula for $f(z; x)$ in terms of the preceding values $f(j; x)$, $j = 1, 2, \ldots, z-1$, for various values of z and x. In conjunction with these expressions, it is necessary to use the following conventions:

$$\begin{aligned}
&\text{A.}\quad P(u; \Delta) = 0 \quad \text{for} \quad u < 0,\\
&\text{B.}\quad R(u; r) = 0 \quad \text{for} \quad u < r,\\
&\text{C.}\quad R(u; 0) = 0 \quad \text{for} \quad u > 0\\
&\qquad\qquad\quad\ \ = 1 \quad \text{for} \quad u = 0.
\end{aligned}$$

Then it will be noted that $f(z; x) = 0$ whenever $z+N < x$, and this is in fact an impossible transition. The first two values of (3) are

$$f(1; x) = e^{\rho}\,R(N+1; x),$$
$$f(2; x) = e^{2\rho}\,[R(N+2; x) - e^{-\rho}\,R(N+1; x)].$$

It is also possible to write each $f(z;x)$ explicitly in terms of the Borel–Tanner coefficients $A(z,x)$. If we define $B(z,x)$ by means of the $(z-x)$th order determinant

$$B(z,x) = (-1)^{z+x} \begin{vmatrix} A(z,z-1) & 1 & 0 & \cdots & 0 \\ A(z,z-2) & A(z-1,z-2) & 1 & \cdots & 0 \\ A(z,z-3) & A(z-1,z-3) & A(z-2,z-3) & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & 1 \\ A(z,x) & A(z-1,x) & A(z-2,x) & \cdots & A(x+1,x) \end{vmatrix} \tag{4}$$

for $z > x$ and $B(z,z) = 1$, then we can write $f(z;x)$ in the form

$$\begin{aligned} f(z;x) &= \sum_{j=1}^{z} e^{\rho j} \rho^{z-j} B(z;j)\, R(N+j;x) \\ &= e^{-\rho N} \rho^{N+z-x} \sum_{j=1}^{z} B(z,j)\, A(N+j,x) \quad (z > 0,\ x \leqslant N). \end{aligned} \tag{5}$$

With a table of $A(z,x)$, the coefficients $B(z,x)$ can be most conveniently calculated from the formula

$$B(z,x) = -\sum_{j=x}^{z-1} A(z,j)\, B(j,x). \tag{6}$$

The first few values of $A(z,x)$ and $B(z,x)$ are given below; to avoid awkward fractions, each value is multiplied by $(z-1)!$

$(z-1)!\,A(z,x)$ A1227 new

z \ x	1 A272	2 A89946	3	4	5	6	7
1	1	—	—	—	—	—	—
2	1	1	—	—	—	—	—
3	3	4	2	—	—	—	—
4	16	24	18	6	—	—	—
5	125	200	180	96	24	—	—
6	1296	2160	2160	1440	600	120	—
7	16807	28812	30870	23520	12600	4320	720

Δ 259334

4!

Δ 4!

$(z-1)!\,B(z,x)$

A1787

z \ x	1	2	3	4	5	6	7
1	1	—	—	—	—	—	—
2	−1	1	—	—	—	—	—
3	1	−4	2	—	—	—	—
4	−1	12	−18	6	—	—	—
5	1	−32	108	−96	24	—	—
6	−1	80	−540	960	−600	120	—
7	1	−192	2430	−7680	9000	−4320	720

Δ A152818

A6043

1293 again ~~6043~~ A6044

To show that, for fixed x, $\Sigma f(z; x) = 1$, consider

$$\sum_{j=1}^{\infty} e^{-\rho j} f(j; x) = \sum_{j=1}^{\infty} R(N+j; x) - \sum_{j=1}^{\infty} f(j; x)\,(1-e^{-\rho j}),$$

which is equivalent to
$$\sum_{j=1}^{\infty} f(j; x) = \sum_{j=1}^{\infty} R(N+j; x).$$

Combining this with (2) gives the value unity provided $x \leqslant N$; in the opposite case the result is clear from (1).

During the red phase, the transition probabilities are simple Poisson expressions. In terms of these and $f(z; x)$, we can write equations involving other probabilities. For example, if p_{xy} is the probability of a transition from x cars at the beginning of green to y cars at the beginning of the following green, we have

$$\left.\begin{aligned} p_{xy} &= P(y-x+N;\ (\alpha+\beta)\lambda) \quad (x > N) \\ &= P(y;\ \lambda\beta) \sum_{j=x}^{N} R(j; x) + \sum_{z=1}^{y} P(y-z;\ \lambda\beta)\, f(z; x) \quad (x \leqslant N). \end{aligned}\right\} \tag{7}$$

Also, letting
$$\pi_n = \Pr\,(n \text{ cars waiting at beginning of green} \mid t = \infty),$$
$$\sigma_n = \Pr\,(n \text{ cars waiting at beginning of red} \mid t = \infty),$$
we have (if these limiting probabilities exist)

$$\sigma_n = \sum_{j=0}^{n+N} f(n; j)\, \pi_j \tag{8}$$

and
$$\pi_n = \sum_{j=0}^{n} P(n-j;\ \lambda\beta)\, \sigma_j. \tag{9}$$

3. Semi-actuated lights

In this section we consider the traffic along a main street into a signalized intersection governed by a light (possibly) actuated by side street traffic. When the light turns green for the main street, it must remain green for a minimum fixed period (the main street minimum). After the expiration of this time the light will change whenever a car arrives on the side street. Then the light will be red on the main street for a fixed period (side street initial), and a further fixed period (side street extension). If any side street vehicles arrive during the first side street extension the main street light remains red for a further side street extension, measured from the time of arrival of the car. This process continues as long as cars arrive on the side street, or until a maximum value (side street maximum) is reached, when it turns green again. We will assume Poisson arrivals on both streets, and use the following notation:

λ = main street input parameter,

μ = side street input parameter,

a = length of side street initial,

b = length of side street extension,

N = number of cars able to clear during main street minimum,

α = length of main street initial,

A = length of side street maximum.

These quantities are fixed constants of the system. In addition, we need to define certain stochastic variables which will be used to characterize the system.

β = length of main street red. Defined over $a+b \leqslant \beta \leqslant A$, with density function $g(u)$,

n = number of cars able to pass through a main street green period. Defined over $N, N+1, N+2, \ldots$, with probability distribution q_n. For simplicity we will call n the number of 'slots' provided by the green phase.

Once explicit forms are found for $g(u)$ and q_n, the problem of determining the various distributions of § 2 is solved. For example, an equation like (3) would only need to be modified to take into account the stochastic nature of the number of slots:

$$f(z;x) = \sum_{n=N}^{\infty} e^{\rho z}[R(n+z;x) - \sum_{j=1}^{z-1} R(z;j)\,f(j;x)]\,q_n.$$

In case the side street maximum is infinite, both of these distributions are easy to obtain. The red phase length distribution is given in a slightly different context by Raff (1951), and was apparently discovered by Garwood (1940). In the notation defined above, it is

$$g(u) = \mu\, e^{-\mu b} \sum_{i=0}^{h-1} (-e^{-\mu b})^i \left\{\frac{[\mu(u-b(1+i))]^{i-1}}{(i-1)!} + \frac{[\mu(u-b(1+i))]^i}{i!}\right\},$$

$$hb \leqslant u \leqslant (h+1)\,b \quad (h = 1, 2, \ldots).$$

The slot distribution can be calculated directly from the Poisson probabilities and turns out to be

$$q_n = 1 - e^{-\alpha\mu} \quad (n = N)$$

$$= (1 - e^{-\mu T})\,e^{-\mu T(n-1)} \quad (n > N),$$

where $\alpha = NT$.

REFERENCES

Borel, E. (1942). Sur l'emploi du théorème de Bernoulli pour faciliter le calcul d'un infinité de coefficients. Application au problème de l'attente à un guichet. *C.R. Acad. Sci., Paris*, **214**, 452–6.

Garwood, F. (1940). An application of the theory of probability to the operation of vehicular-controlled traffic signals. *J. R. Statist. Soc.* (Suppl.), **7**, 65–77.

Haight, F. A. & Breuer, M. A. (1960). The Borel–Tanner distribution. (To appear.)

Raff, M. S. (1951). The distribution of blocks in an uncongested stream of automobile traffic. *J. Amer. Statist. Ass.* **46**, 114–23.

Tanner, J. C. (1953). A problem of interference between two queues. *Biometrika*, **38**, 58–69.