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SIMULATION CONCEPTS FOR STUDYING INCOMPLETE (BUT POTENTIALLY RECURSIVE) SEQUENCES

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Abstract

The recursive behavior of selected complete and incomplete sequences of integers is investigated by simulating "mathematical" systems with conventional linear system diagrams, concepts, and nomenclature. For instance, impulse response in the linear systems simulates the successive appearance of sequence integers.

Practical application arises from the development of algebraic methods for identifying and extending a class of incomplete sequences. A worked example illustrates how a recursion is tested and an incomplete sequence extended indefinitely.

INTRODUCTION. Mathematicians and engineers alike have an interest in properties of sequences of numbers. They frequently find it convenient to express these sequences in one or more of three alternate, yet equivalent, forms. These are (1) a closed expression for the general k^{th} term, (2) a generating function in a variable, say x, such that its series expansion has the kth term as the coefficient of x^k, and (3) a recursion formula (*i.e.*, difference equation) in k with sufficient boundary conditions to initiate the sequence. However, if the starting point for finding (1), (2), or(3) consists of only a finite number of sequence terms, it may be very difficult (or, at worst, impossible) to ascertain the desired forms, (1), (2), and (3).

By simulating the mathematical systems of the sequences by conventional linear systems, the visual, computational, and other aspects of linear system theory are available to help explain and classify certain kinds of sequence behavior. The simulations are often useful in finding ways to complete the mathematical properties of the sequences. At the same time, the mathematics of sequences may provide insights into linear system behavior. This point of view is consistent with today's cross-disciplinary trends.

We will be concerned only with those sequences (a) whose members are either integers or rational fractions and are zero for negative indices, (b) which ultimately exhibit linear recursive behavior, and (c) which are index-invariant. Index-variance and/or nonlinear recursiveness is beyond the scope of the present discussion, and all further reference to recursion will imply sequences governed by (a)-(c) above.

When only a finite portion of sequences are available, it is usually not known whether such sequences are recursively extendable. If they prove to be, and if enough terms are known, the formula for the general term can be calculated and the sequence continued *ad infinitum*.

Assume that the last member of a finite sequence has index k. If members of index k-n through k-1 satisfy an nth order recursion *and* if that recursion can satisfy the known k^{th} (last) member, the finite sequence is said to be *potentially* recursive of order n. Continued application of the recursion then can generate as many successive terms as desired, all of which are recursively compatible with the original finite sequence *including* the kth indexed term. If no n can be found, the discussions are invalid since this implies either insufficient information, or worse yet, that the chosen sequence was part of an index-variant and/or nonlinear recursion.

We will classify sequences with types of linear systems and excerpt ideas from linear system theory to develop the recursion formulas, the generating functions, and the general term for potentially recursive sequences. In so doing, we will present a straightforward method for obtaining the generating function in closed form for potentially recursive sequences and their infinite extensions. From a system point of view, the discrete sequence values are the impulse responses from initially quiescent discrete-time (sampled-data) systems, where the values of variables are assumed to exist only at integral units of t (time) starting with $t = 0$.

SYSTEM THEORY DEVELOPMENT. Discrete linear (sampled-data) systems have linear difference equations as mathematical models. One of the most useful devices for visualizing these systems is the *simulation diagram*. (See Figures 1 and 2 for examples.) Simulation diagrams are oriented flowgraphs whose weighted edges direct and modulate the transmission of values through the system from an input $x(t)$ to an output $y(t)$. For discrete systems, an edge weight is either a coefficient from the difference equation or is a delay operator, z^{-1}, represented physically by a *delayer*. The values existing at the outputs of the delayers are the state variables of the system and are, thereby, components of the state vector. A delay operator acts on time functions to produce replicas delayed by one time unit. The input and output nodes of a simulation diagram appear as *summers* which linearly combine incoming transmissions to the node into a single output from the node.

Although there are indeed physical real-time systems whose behavior closely approximates that predicted by difference equations, we will consider only exact discrete mathematical systems. The variable t need not be restricted to "time" but can simulate spatial or index variation. The output from the mathematical system is a sequence whose values are equally spaced by one unit of t.

The first descriptive classification for our systems is *Mealy* or *Moore*. These terms are borrowed from linear automata theory (see Hill and Peterson [1], pp. 300-305). If the output of a discrete automaton is a direct function of both the input and the state of the system, the system is classified as Mealy. If, however, the output directly depends only on the state of the system, the system is classified as Moore. Even though a Mealy and a Moore system may be called equivalent by virtue of having identical state and output successions, the output of the Moore system will lag that of the Mealy system by at least one unit of time. This feature is evident in mathematical systems as well as in linear physical systems.

The second classification pertains to the number of initial sequence terms which must be accounted for before the nth order recursion of the sequence is established. If the number of terms is n, the sequence and/or its associated system is called *nonsingular*. The n terms in the nonsingular case can also be recognized as the n initial conditions needed for the solution of the associated n^{th} order difference equation. If the number of initial sequence terms exceeds n, the sequence, system, etc. is classified as *singular*. Sequences of this type occur when arbitrary values are chosen as "fillers" for non-physical or non-existent early terms in what eventually becomes an n-recursive sequence (see Liu [2], pp. 68-73). The classification names come from a slight modification of similar terminology in the study of linear sequential circuits (see Gill [3], pp. 58-67). There will be further reference to this classification as it applies to simulation diagrams.

Although there are a variety of forms for equivalent simulation diagrams, the one most suitable for this note is the so-called "shift-register" form shown in Figure 1.

If b_n is nonzero, the system and all corresponding mathematical forms are classified as Mealy. Conversely, if b_n is zero, the classification is Moore. Although substantiation comes later, it can be stated if b_0 and d_1 through d_h are zero, the classification is nonsingular. If either b_0 and/or any of the d's are present, the classification is singular. It can be observed that either a Mealy or Moore system can be singular or nonsingular.

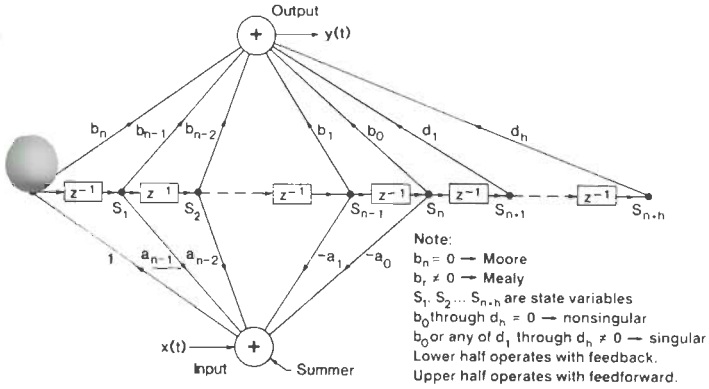

Figure 1. Shift Register Form of General Simulation Diagram

In anticipation of what is to be developed analytically, it is interesting to preview the system behavior with comments on a visual inspection of the simulation diagram. The lower half of the diagram consists of n nested feedback loops wherein state variables existing at discrete values of t are weighted and combined, ready to be reintroduced at the next t value to contribute to the next set of state values. In mathematical as well as physical systems it is in the feedback loops where the natural behavior of the system, in this case, recursion, is established. The upper half of the diagram consists of feedforward paths on which state variables existing at discrete t values are weighted and combined to form the output. When multiplied by the weights of b and d edges, the initial state variables as they are "swept" out of the system form the first $n+h$ terms of mathematical as well as physical systems. Only after this can the nth order recursion appear in the output sequence. It is interesting to note that for given initial state conditions, it is possible (with some degree of patience and care) to "step" a given input sequence through the simulation diagram to produce an output sequence.

While the simulation diagram of Figure 2 defines a specific second order, Mealy, singular system, its analysis is representative of how higher order systems may be treated.

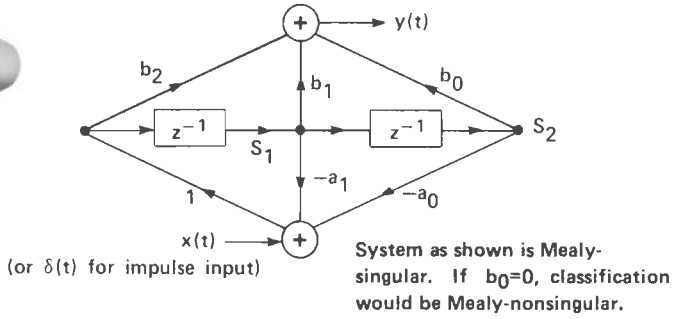

Figure 2. Simulation Diagram for Second Order System

Application of Mason's flowgraph formula [4, 5, 6] to the simulation diagram of Figure 2 produces the *operational* equality

$$\frac{y(t)}{x(t)} = \frac{b_2 + b_1 z^{-1} + b_0 z^{-2}}{1 + a_1 z^{-1} + a_0 z^{-2}} . \tag{1}$$

(A discussion of Mason's formula is outside the scope of this note. The cited references explain how this tool is used, and a rigorous proof is found in Kim and Chien [7].) Cross operation on (1) yields the system's second order difference equation as shown in (2).

$$y(t)+a_1 y(t-1)+a_0 y(t-2)=b_2 x(t)+b_1 x(t-1)+b_0 x(t-2). \tag{2}$$

The right side of (1) is also the conventional system function in z as shown in (3).

$$\frac{Y(z)}{1} = \frac{b_2 z^2 + b_1 z + b_0}{z^2 + a_1 z + a_0} = H(z). \tag{3}$$

INITIAL CONDITIONS FROM IMPULSE INPUTS. Consider expression (2) where the driving function at $t = 0$ is the unit impulse, $\delta(t)$.

$$y(t) + a_1 y(t-1) + a_0 y(t-2) = b_2\delta(t) + b_1\delta(t-1) + b_0\delta(t-2), \tag{4a}$$

$$y(t+2) + a_1 y(t+1) + a_0 y(t) = b_2\delta(t+2) + b_1\delta(t+1) + b_0\delta(t). \tag{4b}$$

In system theory , it is common practice to "trade off" initial conditions for impulse input effects and vice versa. In essence, difference equation (4) can be replaced by its homogeneous form accompanied by appropriate initial or boundary conditions. To establish the initial conditions, the question which must be answered is "What conditions can be found which, in a valid operation, can force the right side of (4) to be identically zero?"

For t-domain use, the right side of (1) is interpreted as an operator. In the z-domain, however, it is the z-transform of the system function, $H(z)$, and follows *algebraic* rules. $H(z)$ is also the response in z to a unit impulse, $\delta(t)$, at t=0. When the input is a unit impulse, the response must equal the $H(z)$ of (3) since the z-transform of a unit impulse is 1. (For convenience the ratio of (3) has been rationalized.) However, application of the one-sided z-transform [8] to (4b) leads to

$$H(z)=\frac{Y(z)}{1}=\frac{b_2 z^2+b_1 z+b_0}{z^2+a_1 z+a_0} +$$

$$\frac{z^2(c_0-b_2\delta(0))+z(c_1+a_1 c_0-b_2\delta(1)-b_1\delta(0))}{z^2+a_1 z+a_0}, \tag{5}$$

where c_0 and c_1 are the first two terms of the output sequence. In order to reconcile (5) with (3), the right ratio in (5) must vanish. Use of the concepts of Distribution Theory and Generalized Functions [9] shows that within the context of our operations, $\delta(t)=1$ when $t=0$ and $\delta(t)=0$ when t is integral and $t\neq 0$. The conditions that the right ratio of (5) vanish are

$$c_0 = b_2 \tag{6a}$$

$$c_1 = -c_0 a_1 + b_1 \tag{6b}$$

Next consider (4a) at successive instants of t. (Recall that c_{-1}, c_{-2}, etc. are all zero and that $y(k)=c_k$.) Equation (4a) becomes

$$c_0=b_2, \quad t=0 \tag{7a}$$

$$c_1=-c_0 a_1+b_1, \quad t=1 \tag{7b}$$

$$c_2=-c_1 a_1-c_0 a_0+b_0, \quad t=2 \tag{7c}$$

$$c_3=-c_2 a_1-c_1 a_0, \quad t=3 \quad \rightarrow \quad y(5)+a_1 y(4)+a_0 y(3)=0 \tag{7d}$$

It is seen that for all discrete $t\geq 3$ the difference equation is homogeneous and, moreover, can be found by direct inspection of the denominator of (3). That the recursion, once established, continues indefinitely is not surprising because we are viewing the direct influence of the feedback polynomial of the simulation diagram, or, what is the same thing, the characteristic equation of the difference equation. The coefficients c_0, c_1, and c_2 are results of the initial state of the system and are identically that output condition forthcoming from the quiescent system under the influence of unit impulse input. If the homogeneous form of the second order difference equation

$$y(t)+a_1 y(t-1)+a_0 y(t-2)=0 \tag{8}$$

If b_0 or any of d_1 through d_h are present, the $h+1$ terms, c_0 through c_h, are functions of b_0 and d_1 through d_h. Even though they contribute to the magnitude of all succeeding terms, the presence of c_0 through c_h forces a delay of $h+1$ units in recursion emergence. The n terms, c_{h+1} through c_{h+n}, become boundary values for the homogeneous recursion which starts with c_{h+n+1}. These are the sequence specifications for the singular classification.

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One observation is in order at this point. To assure uniqueness in difference equation presentation, only homogeneous forms are considered since nonhomogeneous difference equation forms are not necessarily unique. This can be seen from the following difference equations

$$c_n-2c_{n-1}=4, \quad c_0=3 \tag{9a}$$

$$c_n-c_{n-1}=7^{n-1}, \quad c_0=3 \tag{9b}$$

$$c_n-3c_{n-1}+2c_{n-2}=0, \quad c_0=3,\ c_1=10 \tag{9c}$$

all of which generate the output 3, 10, 24, 52, 108, . . .

RECURSION PROPERTIES OF SEQUENCES. While the z-transform was useful in obtaining (6a) and (6b), there is an easier way to continue the c_k's. If in the general form of $H(z)$ (see (3)), the numerator polynomial is divided by the denominator polynomial a series in powers of $1/z$ results. The coefficients are equal in value to the c_k's. However, unless some special arrangement is made, the coefficients, as the index increases, become complicated expressions of the a's, b's, and d's (if any are present), making it very difficult to trace recursions. If the quotient is pre-specified with c_k's already in place, the division process can be forced to show the manner in which c_k's are generated from previous c_{k-} values. This is, of course, the desired recursion. A brief example using a third order system is shown below.

$$
\begin{array}{l}
\qquad\qquad\qquad\qquad c_0 + c_1/z + c_2/z^2 + \ldots \\
z^3 + a_2z^2 + a_1z + a_0 \overline{\big)\, b_3z^3 + b_2z^2 + b_1z + b_0} \\
\qquad\qquad\qquad\qquad c_0z^3 + c_0a_2z^2 + c_0a_1z + c_0a_0 \\
\hline
\qquad\qquad (c_0a_2+b_2)z^2+(-c_0a_1+b_1)z+(-c_0a_0+b_0) \\
\qquad\qquad\qquad\qquad c_1z^2 + c_1a_2z + c_1a_1 \\
\hline
\qquad (-c_1a_2-c_0a_1+b_1)z+(-c_1a_1-c_0a_0+b_0)+(-c_1a_0)/z \\
\qquad\qquad c_2z \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{array}
\tag{10}
$$

By equating the coefficients of like powers in the cancelled leading terms, the buildup of c_k's may be observed. More complete divisions and extension to higher order systems, yield a general pattern. Inspection results for orders three and four reveals the extension pattern.

$$
\begin{aligned}
c_0 &= b_3 \\
c_1 &= -c_0a_2+b_2 \\
c_2 &= -c_1a_2-c_0a_1+b_1 \\
c_3 &= -c_2a_2-c_1a_1-c_0a_0+b_0 \\
c_4 &= -c_3a_2-c_2a_1-c_1a_0 \\
c_5 &= -c_4a_2-c_3a_1-c_2a_0 \\
c_6 &= -c_5a_2-c_4a_1-c_3a_0 \\
c_7 &= -c_6a_2-c_5a_1-c_4a_0 \\
c_8 &= -c_7a_2-c_6a_1-c_5a_0 \\
c_9 &= -c_8a_2-c_7a_1-c_6a_0 \\
c_{10} &= -c_9a_2-c_8a_1-c_7a_0
\end{aligned}
\tag{11a}
$$

$$
\begin{aligned}
c_0 &= b_4 \\
c_1 &= -c_0a_3b_3 \\
c_2 &= -c_1a_3-c_0a_2+b_2 \\
c_3 &= -c_2a_3-c_1a_2-c_0a_1+b_1 \\
c_4 &= -c_3a_3-c_2a_2-c_1a_1-c_0a_0+b_0 \\
c_5 &= -c_4a_3-c_3a_2-c_2a_1-c_1a_0+d_1 \\
c_6 &= -c_5a_3-c_4a_2-c_3a_1-c_2a_0+d_2 \\
c_7 &= -c_6a_3-c_5a_2-c_4a_1-c_3a_0+d_3 \\
c_8 &= -c_7a_3-c_6a_2-c_5a_1-c_4a_0 \\
c_9 &= -c_8a_3-c_7a_2-c_6a_1-c_5a_0 \\
c_{10} &= -c_9a_3-c_8a_2-c_7a_1-c_6a_0 \\
c_{11} &= -c_{10}a_3-c_9a_2-c_8a_1-c_7a_0 \\
c_{12} &= -c_{11}a_3-c_{10}a_2-c_9a_1-c_8a_0 \\
c_{13} &= -c_{12}a_3-c_{11}a_2-c_{10}a_1-c_9a_0 \\
c_{14} &= -c_{13}a_3-c_{12}a_2-c_{11}a_1-c_{10}a_0 \\
c_{15} &= -c_{14}a_3-c_{13}a_2-c_{12}a_1-c_{11}a_0
\end{aligned}
\tag{11b}
$$

If sufficient coefficients, c_0, c_1, c_3, ... , c_k, ... , known to be be part of an ultimately recursive sequence, are available, the first n of the last $n+1$ equations of tabulations (as exemplified by (11) above) can be used to form n simultaneous equations for finding a_{n-1}, a_{n-2}, ... ,a_0. The last equation (the $(n+1)$st) can be used as a substitution check to assure continuation of the recursion. If this test is successful, substitution of the calculated and verified a's in the lower indexed equalities yields the b and d (if any) coefficients. Knowledge of the coefficients establishes $H(z)$, the generating function for the series in $1/z$. By tracing backwards, the simulation diagram and both forms for the difference equation can be found. Through application of the inverse z-transform of $H(z)$, the general sequence term appears.

In starting with a finite sequence, the value of n would probably not be known. This would necessitate judicious choices of successive trial n's (starting with a reasonably large n) and constructing tabulations for each n based on the extended generalizations of (11a) and (11b). We utilized a computer-enhanced version of this technique to sucessfully obtain many solutions including that of the example worked later.

In order to avoid impossible situations and to specify possible choices of n which *might* lead to a successful conclusion, the following conditions are summarized:

A sufficient condition on the minimum number of consecutive terms of a potentially recursive sequence, c_0, c_1, c_2, ... , c_k, . . . , for finding the generating function, $H(z)$, of an nth order, nonsingular, Mealy or Moore, system is that $(3n+1)$ such consecutive terms be available.

Proof: (a) The first n terms, c_0 through c_{n-1}, are needed as boundary conditions for the anticipated homogeneous equation.

(b) The n terms, c_{2n} through c_{3n-1}, are needed in a set of n simultaneous equations for calculating the a cofficients of the denominator of $H(z)$.

(c) The last term, c_{3n}, is used with a check equation to assure the continuance of the recursion noted in (b).

(d) The n terms, c_n through c_{2n-1}, are needed as independent coefficients in the set of simultaneous equations described in (b) above. The bare minimum number of terms required thus total $3n+1$. If there is a possibility of a b_0 term, $3n+2$ terms would be required. However, the system would lose its nonsingular classification.

A sufficient condition on the minimum number of consecutive terms of a potentially recursive sequence, c_0, c_1, c_2, ... , c_k, . . . , for finding the generating function, $H(z)$, of an nth order, singular, Mealy or Moore, system is that $(3n+h+2)$ such consecutive terms be available, where h is the value of the highest d index.

Proof: The singular effects of b_0, d_1 through d_h even though any or all of a_0, d_1 through d_{h-1} are zero, occur before the recursion and the n initialconditions start. Hence, $h+1$ must be added to the nonsingular results to account for the "singular" terms. The total becomes $(3n+h+2)$.

CALCULATION METHODS. It should be quite obvious that in all but the most elementary recursions, hand or even desk calculator computations would be impractical. Machine computation using an appropriate program would be desirable, but the round-off errors in calculations would destroy accuracy when large integers or large rational fractions are manipulated. Fortunately, computer algebra programs such as MACSYMA or muMath [10, 11] are capable of performing symbolic computations and exact computations with extremely large integers or rational fractions. We used a version of muMath on an AT clone to obtain values for the a, b, and d coefficients from sets of sequence terms. The program, written in the muSimp language of muMath, was expediently unsophisticated but was able to test and report on eighth and less order recursions using appropriately sized tables of equations similar to the illustrations of (11). Built in capabilities of muMath calculate $H(z)$ directly once the a coefficients have been found from the n simultaneous equations and checked against the last sequence term. From this point, it is routine to determine the difference equations, simulation diagram, classification of the sequence, system, and general coefficient. Now that it has served its immediate purpose, the program is being revised and compacted for possible later publication.

NUMERICAL EXAMPLE. Suppose that the 19 sequence values, $c_0=2$, $c_1=-4$, $c_2=7$, $c_3=-13$, $c_4=25$, $c_5=-49$, $c_6=97$, $c_7=-193$, $c_8=385$, $c_9=-769$, $c_{10}=1537$, $c_{11}=-3073$, $c_{12}=6145$, $c_{13}=-12289$, $c_{14}=24577$, $c_{15}=-49153$, $c_{16}=98305$, $c_{17}=-196609$, $c_{18}=393217$, are available. The minimum term condition, $3n+1$, suggests that $n=6$ would be the largest trial n permitted. Even if the ultimate recursion were of order 6, no solution could be found unless the system was nonsingular. If the n is, in reality, less than 6, successively smaller trial n's would permit increasing latitude with respect to considering possible b_0 and d coefficients. However, when experimentally searching for real-world solutions, the likelihood of encountering d coefficients in physical or mathematical systems is rare.

In our example, the muMath computations ruled out 6th down through 3rd order solutions. The 2nd order solution using the selected simultaneous equations (12a) and (12b) produced $a_0=2$ and $a_1=3$. These values also satisfy check equation (12c).

$$
\begin{aligned}
&(c_{16}= 98305) && 98305 = 49153a_1 - 24577a_0, && (12a)\\
&(c_{17}= -196609) && -196609 = -98305a_1 + 49153a_0, && (12b)\\
&(c_{18}= 393217) && 393217 = 196609a_1 - 98305a_0. && (12c)
\end{aligned}
$$

Now that a second order recursion is assured and a_1 and a_0 are known, the search for b's and d's can proceed from suitably rearranged equations from the second order version of (11) as shown below.

$$
\begin{aligned}
b_2 &= c_0 \\
b_1 &= c_1 + c_0a_1 \\
b_0 &= c_2 + c_1a_1 + c_0a_0 \\
d_1 &= c_3 + c_2a_1 + c_1a_0 \\
d_2 &= c_4 + c_3a_1 + c_2a_0 \\
&\ldots\ldots\ldots\ldots \\
d_{11} &= c_{13} + c_{12}a_1 + c_{11}a_0.
\end{aligned}
\tag{13}
$$

Through substitution in (13) the d's were found to be zero for this example. However, $b_0 = -1$, $b_1 = 2$, and $b_2 = 2$. Because $b_2 \neq 0$ and $b_0 \neq 0$, the system, sequences, etc., are Mealy and singular. The generating function in z can now be given as

$$\frac{2z^2 + 2z - 1}{z^2 + 3z + 2}. \tag{14}$$

In terms of sequence coefficients, the homogeneous difference equation is

$$c_t + 3c_{t-1} + 2c_{t-2} = 0, \qquad c_1 = -4,\ c_2 = 7, \tag{15}$$

with $c_0 = 2$ as a "filler" consistent with the degree of singularity. Although the general sequence term, good for $t \geq 1$, can be found directly from (15) by either classical or z-transform methods, it is easier to use (14) *after* first dividing denominator into numerator sufficiently to bring out b_0 and all d terms as coefficients of the quotient and then to extract the inverse z-transform of the remainder polynomial over the denominator of (14). For our example, we obtain

$$2 + \frac{1}{z}\left[\frac{-4z^2 - 5z}{z^2 + 3z + 2}\right], \tag{16}$$

where the $1/z$ has been extracted from the bracketed ([ ]) part of (16) so that the inverse z-transform appears correctly shifted with its series starting with c_1/z.

The right side of (16) reduces to

$$\frac{1}{z}\left[\frac{-z}{z+1} + \frac{-3z}{z+2}\right] \tag{17}$$

from which the inverse z-transform yields the general coefficient of the recursive part (properly index adjusted) as

$$c_t = (-1)^t + \frac{3}{2}(-2)^t. \tag{18}$$

CONCLUSIONS. Under the assumption that linear recursive sequences of integers are the outputs of hypothetical "mathematical" systems, it was demonstrated that conventional linear systems could simulate such mathematical systems. This made it possible for linear system theory to analyze and predict the behavior of the mathematical systems. To implement the simulation, system theory nomenclature was used to describe the characteristics of the sequence of integers.

By taking advantage of special properties of linear recursive sequences, a method for constructing the simulating system was devised. As an additional consequence, desired features of the sequence itself became evident. However, success with the method depended on availability of sufficient sequence data and the use of a computer algebra system to insure exact integer manipulation.

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