Indexed

~~to NJAS~~

1181
–1185

GENERATION OF A CLASS OF PERMUTATIONS ASSOCIATED WITH COMMUTING FUNCTIONS

by

William M. Boyce

MATHEMATICAL ALGORITHMS
Vol. 2 (1967)
pp. 19-26

(MA4)

GENERATION OF A CLASS OF PERMUTATIONS ASSOCIATED
WITH COMMUTING FUNCTIONS

William M. Boyce

I. INTRODUCTION

This paper deals with the systematic generation and classification of a class of permutations (Baxter permutations) which arose in the study of commuting functions. The principal question concerning commuting functions was raised in 1954 by Eldon Dyer. It may be stated as follows: let f and g be continuous functions mapping the unit interval into itself which commute under functional composition, that is, $f(g(x)) = g(f(x))$ for all x. Must f and g have a common fixed point, meaning a point z such that $f(z) = z = g(z)$? This question was answered in the negative by the author's construction of a "counterexample" [4], making use of the results presented here. (J. P. Huneke has independently constructed negative examples [5], using a different technique.) Under a mild finiteness condition, a pair of commuting functions induces a Baxter permutation; conversely, a Baxter permutation can often be used to define a pair of commuting functions which induce it. The significance of this should be judged in the light of Baxter and Joichi's statement that [3] ". . . among the primary difficulties encountered in attempting to verify the conjecture [of a common fixed point] in more general cases is the lack of a plentiful supply of examples to investigate." By systematically generating Baxter permutations and attempting to extend them to commuting functions, the author discovered a pair of commuting functions with no common fixed point.

In what follows, N always denotes an odd natural number, I_N the set of natural numbers through N, O_N the odd elements of I_N, E_N the even elements of I_N, and P is permutation P: $I_N \rightarrow I_N$.

DEFINITION. P: $I_N \rightarrow I_N$. is a Baxter permutation of order N if and only if

(1) $P(O_N) = O_N$, $P(E_N) = E_N$;
(2) if $P(n)$ is between $P(2i)$ and $P(2i+1)$, then $n \geq 2i$;
(3) if $P(n)$ is between $P(2i-1)$ and $P(2i)$, then $n \leq 2i$.

The intervals $[2i, 2i+1]$ are called up-intervals and the intervals $[2i-1,2i]$ down-intervals.

The relationship between commuting functions and Baxter permutations is given in the following theorem [1].

THEOREM. (Baxter) Let f and g be continuous functions mapping the unit interval into itself which commute, let $h(x) = f(g(x)) = g(f(x))$, let H' be the set of fixed points of h, and suppose that H' is finite and contains neither endpoint of the unit interval. Let H_1 be the set of points where $h(x)-x$ goes from negative to positive (up-crossing points), H_2 the set of points where $h(x)-x$ goes from positive to negative (down-crossing points), and H_3 the set of points where the graph of h touches the diagonal but does not cross it. Let $H = H_1 \cup H_2$. Then

(1) $f(H') = H'$, $f(H_1) = H_1$, $f(H_2) = H_2$, so f is a permutation of H, and g is the inverse permutation;

(2) elements of H_1 and H_2 alternate, with the first and last elements in H_2, so that if the elements of H are denoted $t_1, t_2, \ldots, t_N$, in order, the odd-numbered are in H_2 and the even-numbered in H_1, then

(a) if $f(t_n)$ is between $f(t_{2i})$ and $f(t_{2i+1})$, then $n \geq 2i$;

(b) if $f(t_n)$ is between $f(t_{2i-1})$ and $f(t_{2i})$, then $n \leq 2i$;

(3) $P: I_N \to I_N$ given by $P(i) = j$ if and only if $f(t_i) = t_j$ is a Baxter permutation.

Note that the original significance of the terms "up-interval" and "down-interval" were that they described whether the graph of h was above or below the diagonal on the interval.

If we have an x such that $f(x) = x$ and $h(x) = x$, then $g(x) = g(f(x)) = h(x) = x$, so that if f has a fixed point in H', f and g have a common fixed point. In particular, if the Baxter permutation induced by f and g has a fixed point, then f and g have a common fixed point. For these permutations the answer to Dyer's question is obvious, so they will be called trivial Baxter permutations.

The steps taken by the author in his search for a counterexample were for each successive N to generate the set of Baxter permutations of order N, distinguish between the trivial and non-trivial ones, and give special attention to each non-trivial case. The object of the special attention was to attempt to establish whether the permutation could be induced by commuting functions and, if so, whether the functions must have a common fixed point (which would then be a point of H_3). To reduce the amount of special attention required, Baxter permutations were divided into equivalence classes relative to the common fixed point question. To reduce the amount of labor in generating and equivalencing permutations, the author discovered theorems and algorithms which generate Baxter permutations efficiently and wrote a computer program incorporating them. The following section presents these results.

II. THEORY

The first theorem is an aid in generating the permutations. The "inverse" part was proved by Baxter and Joichi in [2] ; the "reflection" parts are left to the reader.

THEOREM 1. If P: $I_N \to I_N$ is a Baxter permutation, then P^{-1} is also. Let U: $I_N \to I_N$ be defined by $U(i) = N + 1 - i$. Then UP, PU, and UPU are also Baxter permutations.

Corollary. P^{-1} and UPU are Baxter permutations which have a fixed point if and only if P does. An equivalence class under these relationships consists of P, P^{-1}, UPU, and $UP^{-1}U$, and may contain one, two, or four members due to possible duplication.

To see why this definition of equivalence class is chosen, suppose that f and g are commuting continuous functions on the unit interval which induce the permutation P. Then by simply redesignating g as f and f as g, the permutation induced will be changed to P^{-1}; and if we define $u(x) = 1-x$, then ufu and ugu will commute and induce the permutation UPU. In either case the property of having or not having a common fixed point is preserved.

The following theorem provides the greatest savings in generating Baxter permutations. It states that the action of P on the even elements is determined by its action on the odd elements. Thus only $((N+1)/2)!$ permutations must be considered for N, rather than N!

THEOREM 2. Let P be a Baxter permutation of order N and let j be even. Then

(1) if $P(j+1) > P(j-1)$, then $P(j)$ is the least element of E_N which is greater than $P(j-1)$ and not in $P(I_{j-1})$;

(2) if $P(j+1) < P(j-1)$, then $P(j)$ is the greatest element of E_N which is less than $P(j-1)$ and not in $P(I_{j-1})$.

Proof. For (1), since j is even, $[j,j+1]$ is an up-interval, so we cannot have $P(j-1)$ between $P(j)$ and $P(j+1)$. Thus $P(j) > P(j-1)$. The interval $[j-1,j]$ is a down-interval, so for any $n = P(i)$ between $P(j-1)$ and $P(j)$, we must have $i < j$, or n in $P(I_{j-1})$. Thus $P(j)$ is the least element of E_N greater than $P(j-1)$ and not in $P(I_{j-1})$. Part (2) is the statement of part (1) for the Baxter permutation UP.

Corollary. P: $E_N \to E_N$ is determined by P: $O_N \to O_N$, so there are at most $((N+1)/2)!$ Baxter permutations of order N.

Proof. Given P: $O_N \to O_N$, P(2) is determined by P on O_N, P(4) is determined by P on O_N and P(2), et cetera, so that P: $E_N \to E_N$ is inductively determined.

The next theorem is not strictly necessary for the generation of the permutations but is stated for completeness.

DEFINITION. A subset C of $S \subset I_N$ is called connected-rel-S if no element of S not in C lies between two elements of C. A component-rel-S of X is a maximal connected-rel-S subset of X.

THEOREM 3. Given any permutation Q: $O_N \to O_N$, the extension of Q to E_N given by Theorem 2 and the Corollary is well-defined and results in a permutation satisfying parts (1) and (3) of the definition--that is, at most the up-intervals are violated. Each component-rel-I_N of $Q(I_{2i})$ contains equal numbers of odd and even elements.

Proof. Let us prove the last statement first, by induction. $Q(I_2)$ consists of $Q(1)$ and either $Q(1)+1$ or $Q(1)-1$, depending on whether $Q(3) > Q(1)$ or $Q(3) < Q(1)$. Thus $Q(I_2)$ has one component with one odd element and one even element. Suppose now that each component of $Q(I_{2i})$ has equal numbers of odd and even elements. Assume $Q(2i+3) > Q(2i+1)$. There are k odd elements and k+1 even elements between $Q(2i+1)$ and $Q(2i+3)$, so since neither $Q(2i+1)$ nor $Q(2i+3)$ is in $Q(I_{2i})$, there are at most k odd elements and k even elements between $Q(2i+1)$ and $Q(2i+3)$ which are in $Q(I_{2i})$. Thus there is an even element left to be chosen as $Q(2i+2)$. If $Q(2i+1)+1$ is not in $Q(I_{2i})$, then $Q(2i+2) = Q(2i+1)+1$, and $\{Q(2i+1), Q(2i+2)\}$ is a connected set. If $Q(2i+1)+1$ is in $Q(I_{2i})$, then it must lie in a component of $Q(I_{2i})$ with equal numbers of odd and even elements, so that $Q(2i+1)+2, \ldots, Q(2i+1)+2n$ are in $Q(I_{2i})$, and $Q(2i+1)+2n+1$ is not in $Q(I_{2i})$ and is even. Then $Q(2i+2) = Q(2i+1)+2n+1$, so $Q(2i+1)$ and $Q(2i+2)$ are in the same component-rel-I_N of $Q(I_{2i+2})$. In either case $Q(2i+1)$ and $Q(2i+2)$ lie in a connected subset of $Q(I_{2i+2})$, so each component of $Q(I_{2i+2})$ will have equal numbers of odd and even elements. The proof for $Q(2i+3) < Q(2i+1)$ is exactly the same, so the assertion is proved by induction. To show that the extension of Q from O_N to I_N is well-defined, it suffices to show that there is always an even element between $Q(2i+1)$ and $Q(2i+3)$ which is not in $Q(I_{2i})$. But this follows from the nature of the components of $Q(I_{2i})$, since there are k odd and k+1 even elements between $Q(2i+1)$ and $Q(2i+3)$, so at most k of the even ones can be in $Q(I_{2i})$. Thus the extension is well-defined. To show that Q: $I_N \to I_N$ satisfies parts (1) and (3) of the definition of Baxter permutation, only part (3) must be verified; so it must be shown that if $Q(n)$ is between $Q(2i-1)$ and $Q(2i)$, then $n \le 2i$. Every even element n between $Q(2i-1)$ and $Q(2i)$ must be in $Q(I_{2i-2})$, or else it would have been chosen for $Q(2i)$ in preference to the value chosen. But since there are an even number of elements between $Q(2i-1)$ and $Q(2i)$, and because of the nature of the components of $Q(I_{2i-2})$, for each odd element between $Q(2i-1)$ and $Q(2i)$ there must be a similar even element. Since this cannot be, part (3) will always be satisfied.

Corollary. If $Q\colon O_N \to O_N$ is a permutation, let Q^x denote the extension of Q to I_N as given by Theorem 2 and its corollary. Tnen Q^x is a Baxter permutation if and only if $Q^x = (QU)^xU$.

Proof. $(QU)^xU$ is the extension of Q to I_N one gets by running Theorem 2 and its corollary "backwards," first defining Q(N-1) (N-1 being even), then defining Q(N-3) in terms of $Q\colon O_N \to O_N$ and Q(N-1), et cetera. $(QU)^xU$ will satisfy parts (1) and (2) of the definition. If $Q^x = (QU)^xU$, then Q^x satisfies all three parts of the definition and is thus a Baxter permutation. On the other hand, if P is a Baxter permutation, then PU is also; thus P is uniquely determined by either $P\colon O_N \to O_N$ or $PU\colon O_N \to O_N$.

Before the author discovered the method of direct generation of Baxter permutations on O_N, the preceding corollary proved useful in separating the permutations of O_N which could be extended to Baxter permutations from those which could not.

The next theorem characterizes the action of Baxter permutations on the odd elements in a manner ideally suited for computer generation of the permutations.

THEOREM 4. Let $P\colon I_N \to I_N$ be a Baxter permutation and let j be odd.

(1) If $P(j) < P(j-2)$, let m be the least element of the component-rel-O_N of $P(O_{j-2})$ containing $P(j-2)$, and let $M_2(j) = m-2$. Then $M_2(j)$ is odd and not in $P(O_{j-2})$. Let $M_1(j)$ be the least element of the component-rel-O_N of $O_N \setminus P(O_{j-2})$ which contains $M_2(j)$. Tnen $M_1(j) \le P(j) \le M_2(j)$.

(2) If $P(j) > P(j-2)$, let m be the greatest element of the component-rel-O_N of $P(O_{j-2})$ which contains $P(j-2)$, and let $M_3(j) = m+2$. Then $M_3(j)$ is odd and not in $P(O_{j-2})$. Let $M_4(j)$ be the greatest element of the component-rel-O_N of $O_N \setminus P(O_{j-2}0$ which contains $M_3(j)$. Then $M_3(j) \le P(j) \le M_4(j)$.

Proof. For part (1),by definition $P(j) \le M_2(j)$. Suppose that $P(j) < M_1(j)$. Then there is an element n of $P(O_{j-2})$ such that $P(j) < n < M_1(j)$, since otherwise P(j) and $M_1(j)$ would be in the same component-rel-O_N of $O_N \setminus P(O_{j-2})$. If $P(j-1) > n$, then n would be between P(j-1) and P(j) but $P^{-1}(n) < j-2$; but since j-1 is even and j odd, this violates part (2) of the definition of a Baxter permutation. If $P(j-1) < n$, then $M_2(j)$ is between P(j-2) and P(j-1), with $P^{-1}(M_2(j)) > j$. Since j-2 is odd and j-1 even, this violates part (3) of the definition. Part (2) of the theorem is proved similarly.

Corollary. The following process will generate all Baxter permutations of order N: pick a value P(1), then a value P(3) such that $P(3) \neq P(1)$. Define $M_1(5)$, $M_2(5)$, $M_3(5)$, and $M_4(5)$. Choose P(5) such that either $M_1(5) \le P(5) \le M_2(5)$ or

$M_3(5) \leq P(5) \leq M_4(5)$, except that if $M_2(5) = -1$ or $M_3(5) = N+2$, then $P(5)$ must be chosen from the other interval. Continue in this manner for $P(7)$, $P(9)$, et cetera, until $P\colon O_N \to O_N$ has been defined. Then extend P to E_N by the rules of Theorem 2 and its corollary.

THEOREM 5. Every permutation generated in the manner described in Theorem 4 and its corollary will be a Baxter permutation.

Proof. By Theorem 3, the permutation P will satisfy parts (1) and (3) of the definition of Baxter permutation, so it remains to prove part (2) is satisfied. Suppose not; then let i be the least integer for which there is an element $n < 2i$ such that $P(n)$ is between $P(2i)$ and $P(2i+1)$--we may assume that n is also the greatest of its kind. Suppose now that $P(2i) < P(2i+1)$, so that $P(2i-1) < P(2i)$. If there were a k such that $n < k < 2i$ and $P(k)$ and $P(k+1)$ lay on opposite sides of the interval $[P(2i), P(2i+1)]$, then both $P(n)$ and $P(2i)$ would lie between $P(k)$ and $P(k+1)$, which would make $[k,k+1]$ simultaneously an up-interval and a down-interval; but since we assumed that 2i was the least element for which such ambiguity could occur, no such k can exist. Thus we must have $P(n+1) < P(2i)$, for if $P(n+1) > P(2i+1)$, by the preceding argument about k there is no way to get back down to $P(2i-1)$. Since $P(2i)$ is between $P(n)$ and $P(n+1)$, $[n,n+1]$ must be an up-interval, so n must be even. This implies that $P(n)$ must be the least element of $P(I_{2i-2}) \cap [P(2i), P(2i+1)]$. Thus $P(2i)+1$ is odd and is not in $P(O_{2i-1})$, so $M_3(2i+1) \leq P(2i)+1$. The component-rel-I_N of $P(I_{2i})$ which contains $P(n)$ must contain at least one odd element, in particular $P(n)+1$, since $P(n)$ is the least element of the component. Thus $M_4(2i+1) < P(n)+1 < P(2i+1)$. But this violates the conclusions of Theorem 4. A similar argument may be given when $P(2i) > P(2i+1)$. Thus if a permutation is defined on O_N to agree with Theorem 4, and extended to I_N to agree with Theorem 2, then it is a Baxter permutation.

III. IMPLEMENTATION

Theorems 2 and 4 have been implemented in a FORTRAN computer program so as to generate all Baxter permutations of order N. The equivalence classes containing P and UP are generated in the same cycle. The order in which they are generated is lexicographical, based on the "least" member of the pair of equivalence classes. First $P\colon O_N \to O_N$ is generated, using the algorithm of Theorem 4. Then the other permutations P^{-1}, UPU, $UP^{-1}U$, UP, $P^{-1}U$, PU, and UP^{-1} are computed from P. P is lexicographically compared with each of the seven permutations; if any is less than P, then the pair of equivalence classes has been generated earlier, so the group is rejected. If P is lexicographically least, P and UP are checked to see if they have a fixed point. Then duplications among the permutations are eliminated, and the resulting equivalence classes (or class) are recorded.

The most complicated routine is the computation of P: $O_N \to O_N$. As written, this routine includes two explicit DO loops, for P(1) and P(3), and a general-purpose DO which cycles the current value of P(2i+1) for i > 1. After the choice of a value for P(2i+1), the $M_1(2i+3)$ are computed and a vector of admissable values for P(2i+3) established. Then the least admissable value is chosen, and the process continues. The algorithm given in Theorem 4 has no "dead ends" except when P(N) is defined. Then the process is kicked back up to the lowest level for which an admissable value remains.

The following tables gives some of the quantitative results.

N	((N+1)/2)!	#Baxter p.	Ratio	Non-trivial	Eq. cl. of NT
3	2	2	1.00	0	0
5	6	6	1.00	2	1
7	24	22	.92	2	1
9	120	92	.77	22	7
11	720	422	.59	66	21
13	5040	*	*	*	112
15	40320	10754	.27	1694	456
17	*	*	*	*	*

*(To be supplied later.)

The counterexample found by the author was developed from a permutation with N = 13, namely $\begin{pmatrix} 1 & 3 & 5 & 7 & 9 & 11 & 13 \\ 11 & 9 & 1 & 3 & 7 & 13 & 5 \end{pmatrix}$. There is one permutation with N = 11 which could probably be developed into a "least" counterexample.** For all permutation with N ≤ 9, and for all other permutations for 11 and 13 except the equivalence classes mentioned, if f and g commute and induce them, then f and g must have a common fixed point.

** $\begin{pmatrix} 1 & 3 & 5 & 7 & 9 & 11 \\ 11 & 7 & 3 & 1 & 5 & 9 \end{pmatrix}$

BIBLIOGRAPHY

[1] Glen Baxter. On fixed points of the composite of commuting functions, Proc. Amer. Math. Soc. 15 (1964), 851-855.

[2] ________ and J. T. Joichi. On permutations induced by commuting functions, and an imbedding question, Math. Scand. 13 (1963), 140-150.

[3] ________ and ________. On a class of commuting functions, Nieuw Arch. voor Wisk. (3) 12 (1964), 126-136.

[4] William M. Boyce. Commuting functions with no common fixed point, Trans. Amer. Math. Soc., (to appear).

[5] J. P. Huneke. Two counterexamples to a conjecture on commuting continuous functions of the closed unit interval, Not. Amer. Math. Soc. 14 (1967), 284, abstract 67T-231.

Tulane University, New Orleans, and
NASA Manned Spacecraft Center, Houston.