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PARTITIONS IN SQUARES

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Abstract.

In this paper we study primarily partitions in different squares. A complete characterization of the least number of terms needed in different cases is given. The asymptotic number of partitions in squares and in different squares is deduced by use of numerical results obtained from extensive computer runs. Some other related problems are also discussed.

Introduction.

We consider a set $T=\{t_1, t_2, \ldots\}$ where t_i are integers and $1 \leqq t_1 < t_2 < t_3 < \ldots$. A *partition* of the positive integer N is then defined as a representation $N=\sum_i n_i t_i$ with integer coefficients $n_i \geqq 0$. *Compositions* are partitions where different permutations are also counted. Further rules may be added, but we restrict ourselves to the case when repetitions are not allowed (i.e. $n_i=0$ or 1). We denote by $p(n)$ and $q(n)$ the number of partitions with and without repetition, and by $r(n)$ and $s(n)$ the number of compositions with and without repetition. Introducing the generating functions $P(x)=\sum p(n)x^n$, $Q(x)$, $R(x)$ and $S(x)$ being defined analogously, we have:

$$P(x) = \prod_{t \in T} (1-x^t)^{-1} \tag{1}$$

$$Q(x) = \prod_{t \in T} (1+x^t)\,. \tag{2}$$

From the simple observation that the number of compositions starting with t_k is $r(n-t_k)$ we get

$$r(n) = \sum r(n-t_k) \tag{3}$$

with summation over k as long as $n-t_k \geqq 0$; $r(0)=1$. The characteristic equation of (3) is

$$\varphi(\mu) = 1-\mu^{t_1}-\mu^{t_2}-\ldots = 0, \quad \mu = \lambda^{-1}\,. \tag{4}$$

Comparing coefficients we obtain

$$R(x) = \varphi(x)^{-1}\,. \tag{5}$$

The real roots of (4) lie in the interval $-1<\mu<1$ and we find the solution of (3):

$$r(n) = \sum C_i \mu_i^{-n}, \quad C_i = -[\mu_i \varphi'(\mu_i)]^{-1}\,. \tag{6}$$

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In general, the contributions from the non-real roots are small and appear as noise.

We refrain from discussing the functions $s(n)$ and $S(x)$.

Partitions in squares.

From now on we define $t_n = n^2$, $n = 1, 2, 3, \ldots$. We then have the generating function

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$$(7) \quad P(x) = \{(1-x)(1-x^4)(1-x^9)\ldots\}^{-1} = 1 + x + x^2 + x^3 + 2x^4 + 2x^5 + 2x^6 + 2x^7 + 3x^8 + 4x^9 + 4x^{10} + 4x^{11} + 5x^{12} + 6x^{13} + \ldots$$

The coefficients $p(n)$ have been computed up to $n = 25{,}000$. In Table I we give some values of $p(n)$ and $q(n)$; as can be expected the values of $p(n)$ grow much faster than $q(n)$ for increasing values of n.

Table I. *Number of quadratic partitions with and without repetition.*

n	$p(n)$	$q(n)$
100	1116	3
500	9653806	109
1000	3.9984684 (9)	1269
2000	9.7316068 (12)	27526
5000	9.8285000 (18)	8835288
10000	1.0659872 (25)	3.2960898 (9)
25000	4.1689369 (35)	1.1491431 (14)

Assuming an asymptotic representation of the form $p(n) \sim Cn^{-\alpha} \exp(\beta n^{\gamma})$ we found that γ must be very close to 1/3. Taking $\gamma = 1/3$ exactly we obtain $\beta = 3.30716$, $\alpha = 1.16022$ and $C^{-1} = 18.79656$. The values of $q(n)$ fluctuate considerably and we have only been able to establish that the main factor in $q(n)$ seems to be of the order $\exp(\delta x^{1/3})$ with δ approximately equal to 1.4.

Partitions in different squares.

Our primary task will be to distribute the natural numbers into different classes C_1, C_2, C_3, C_4, C_5, C_6 and C_E. These classes are defined as follows. A number $n \in C_k$, $k = 1(1)6$, if there exists a representation of n as the sum of k different squares, but no representation using less than k different squares. The exceptional class C_E contains the numbers which cannot be written as a sum of different squares at all. We introduce the generating function

$$Q(x,z) = (1+zx)(1+zx^4)(1+zx^9)(1+zx^{16})\ldots = 1 + z(x + x^4 + x^9 + \ldots)$$
$$+ z^2(x^5 + x^{10} + x^{13} + x^{17} + x^{20} + x^{25} + x^{26} + \ldots) + z^3(x^{14} + x^{21} + x^{26} + x^{29} + \ldots)$$
$$+ z^4(x^{30} + x^{39} + \ldots) + z^5(x^{55} + \ldots) + \ldots = 1 + zQ_1(x) + z^2 Q_2(x) + \ldots$$

We see that e.g. x^{25} is present in both $Q_1(x)$ and $Q_2(x)$, and that x^{26} is present in both $Q_2(x)$ and $Q_3(x)$. From the definition it is clear that $25 \in C_1$ and $26 \in C_2$.

We found that $q(n)=0$ in 31 cases when $n \leqq 128$. We first prove the following

THEOREM. *Assuming that every integer n in the interval $s \leqq n \leqq t$, where $t = s + [\sqrt{(s+2)}+2]^2$, $s>0$, can be written as a sum of unequal squares, then all integers $\geqq s$ have the same property. Here, as usual, $[\cdot]$ denotes the integer part.*

PROOF. Define $u=[(t-s+1)/2]^2+s$ and consider the interval $t<m\leqq u$. Then every integer m in this interval can be written x^2+n where $x=[\sqrt{(m-s)}]$ with $s\leqq n\leqq t$ (implying that n can be partitioned) because

1) $n = m-x^2 \geqq m-(\sqrt{(m-s)})^2 = s$,

2) $n = m-x^2 \leqq m-(\sqrt{(m-s)}-1)^2 = s-1+2\sqrt{(m-s)}$

$$\leqq s-1+2\sqrt{\left\{\left(\frac{t-s+1}{2}\right)^2+s-s\right\}} = t\ .$$

Further $x^2>n$ since

$$x^2-n = 2x^2-m = 2[\sqrt{(m-s)}]^2-m > 2(\sqrt{(m-s)}-1)^2-m$$
$$= (\sqrt{(m-s)}-2)^2-s-2 > (\sqrt{(t-s)}-2)^2-s-2 = 0\ .$$

In this way a partition in the smaller interval implies a partition in the larger interval. It is easy to see that this larger interval meets the requirements of the theorem. Hence we have a guarantee that a new step can be taken, and so the whole process can be repeated indefinitely.

COROLLARY. *All integers >128 can be written as sums of unequal squares.*

REMARK. Using a similar technique we can prove that every large integer can be written as a sum of unequal pth powers, if there is a sufficiently large interval in which every integer has this property.

We also found empirically that the class C_6 only contains two numbers, viz. 124 and 188 ($124=1^2+2^2+3^2+5^2+6^2+7^2$; $188=1^2+2^2+3^2+5^2+7^2+10^2$). Hence, in our computations we could limit ourselves to computing Q_i, $i=1(1)5$. Calculations carried up to $n=100{,}000$ show that all numbers >188 can be written as sums of not more than 5 different squares. The results obtained are displayed below.

$$C_E = \{2,3,6,7,8,11,12,15,18,19,22,23,24,27,28,31,32,33,$$
$$43,44,47,48,60,67,72,76,92,96,108,112,128\}$$

$$C_6 = \{124,188\}$$

$C_1 = \{n \mid n = m^2,\ m > 0\}$

$C_2 = \{n \mid n = a^2 + b^2,\ a, b > 0,\ a \neq b,\ n \neq c^2\}$

$C_4 = \{A \cdot 4^n,\ n \geqq 0\};\ A = 51, 57, 99, 102, 123, 163, 177, 187,$
$267, 627$ and $8k+7,\ k = 4, 7, 8, 9, 10, 11,$ and $k \geqq 13$.

$C_5 = \{A \cdot 4^n,\ n \geqq m\};\ A = 2, 3, 6, 11, 18, 19, 22, 27, 33, 43, 67$
and $8k+7,\ k = 0, 1, 2, 3, 5, 6, 12$
(i.e. $A = 2, 3, 6, 7, 11, 15, 18, 19, 22, 23, 27, 31, 33, 43, 47,$
$55, 67, 103$); m is given in the table below.

A	2	3	6	7	11	15	18	19	22	23	27	31	33	43	47	55	67	103
m	4	3	3	3	2	2	2	2	1	2	2	2	1	1	2	0	1	0

$C_3 = \{n \mid n > 0,\ n \notin C_1, C_2, C_4, C_5, C_6, C_E\}$.

Adding the values of m above we obtain 33, corresponding to the 31 elements in C_E and the 2 elements in C_6.

The number of elements in $N < n \leqq 4N$ $(N \geqq 188)$ belonging to the classes $C_1 - C_5$ is approximately (total number $= 3N$):

C_1 $\sqrt{N}$
C_2 $KN(4/\sqrt{(\ln 4N)} - 1/\sqrt{\ln N})$ (according to Ramanujan [2] $K = 0.764\ldots$)
C_4 $N/2$
C_5 18 (exactly)
C_3 the rest.

In the interval $250 < n \leqq 1000$ there are 16, 211, 377, 128 and 18 numbers belonging to C_1, C_2, C_3, C_4 and C_5 respectively, while the theoretical results are 16, 209, 382, 125 and 18. The amount of numbers belonging to C_2 decreases very slowly and is still 5 per cent when $N = 10^{100}$(!).

The structure $A \cdot 4^n$ of the numbers in C_4 can be explained to some extent if we remember the well-known theorem established by Jacobi: The number $r_4(n)$ of representations of an integer n as a sum of four squares is equal to 8 times or 24 times the sum of the odd divisors of n depending on n being odd or even.

Note that 0, $\pm n_k$ and permutations are all allowed as well as several identical squares. This means that the number of representations is counted in accordance with the generating function

$$\left(\sum_{k=-\infty}^{\infty} x^{k^2}\right)^4 = \sum_{n=0}^{\infty} r_4(n) x^n .$$

If n is even, then the number of representations is the same for $4n$, and since $a^2 + b^2 + c^2 + d^2 = n$ has an exact counterpart in $(2a)^2 + (2b)^2 + (2c)^2 + (2d)^2 = 4n$, the number of partitions in 4 *different* squares must be the same in both cases. However, we refrain from a more thorough discussion.

Compositions in squares.

The difference equation to be satisfied is

$$r(n) = r(n-1)+r(n-4)+r(n-9)+\ldots,\quad r(0) = 1$$

with the characteristic equation

$$\mu+\mu^4+\mu^9+\mu^{16}+\ldots = 1 \qquad (\mu=\lambda^{-1})\,.$$

The only real solution is $\mu=0.7053466815$ or $\lambda=1.417742546$ giving $r(n)\sim C\lambda^n$ with $C=(\mu+4\mu^4+9\mu^9+\ldots)^{-1}=0.4654211338$. Finally, the generating function is

$$(1-x-x^4-x^9-x^{16}-\ldots)^{-1} = 1+x+x^2+x^3+2x^4+3x^5+4x^6+5x^7+7x^8$$
$$+11x^9+16x^{10}+22x^{11}+30x^{12}+\ldots$$

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REFERENCES

1. G. E. Andrews, *The Theory of Partitions*, Encyclopedia of Mathematics and its Applications, Addison–Wesley Publishing Company (1976).
2. G. H. Hardy, *Ramanujan: Twelve Lectures on Subjects Suggested by His Life and Work*, Cambridge University Press, (1940), 60–64.

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