Theorem: Define
\[ f(n) = \sigma(n) - n \]
where \( \sigma(n) \) is the sum of all divisors of \( n \) except \( n \) itself.
Then the range of \( n \) has density at most \( 47/48 \).

Notation: \( a*b \) is multiplication, \( a**c \) is exponentiation.

Proof: consider integers \( n \) such that \( f(n) \) is divisible by 12,
and \( f(n)<N \), sorted by the residue \( n \mod 12 = \sigma(n) \mod 12 \).

\[ n \mod 12 = 0 \]
Then \( \sigma(n)/n \geq (7/4)(4/3) = 7/3 \)
\( N > \sigma(n)-n > 4n/3 \)
\( n < 3N/4 \), \( n \) divisible by 12.

The number of such \( n \) is at most \( (1/12)(3N/4) = N/16 \).

\[ n \mod 12 = 2, 6, \text{ or } 10. \]
Since each odd prime occurring to an odd power contributes
at least a factor of 2 to \( \sigma(n) \), and \( \sigma(n) \) has only
one factor of 2, we have \( n = 2*p*(s**2) \), where \( p \) is a prime.
Such numbers have density 0.
Also \( N > \sigma(n)-n > (3/2)n - n = n/2 \), so \( n<2N \).
So the number of such \( n \) is \( o(2N) = o(N) \).

\[ n \mod 12 = 4 \text{ or } 8. \]
No primes of the form \( 6k-1 \) occur to an odd power in \( n \)
(or else 3 would divide \( \sigma(n) \)). Such \( n \) have 0 density.
Further, \( N > \sigma(n)-n > (7/4)n - n = 3n/4 \), so \( n<4N/3 \).
Again the number of such \( n \) is \( o(4N/3) = o(N) \).

\[ n \mod 12 = 1, 5, 7, \text{ or } 11. \]
No odd primes occur to an odd power (and 2 doesn’t occur at all).
\( n = s**2 \). Further no prime \( p \) of the form \( 6k+1 \) occurs to
the FIRST power in \( s \). (If \( s \) is divisible by such \( p \), then it’s
divisible by at least \( p**2 \).) \( N>s \). The number of \( s<N \) of
with no prime \( p=6k+1 \) occurring with exponent exactly 1, is
\( o(N) \).

\[ n \mod 12 = 3 \text{ or } 9. \]
\( n \) is divisible by 3, so \( N > \sigma(n)-n > (4/3)n - n = n/3 \),
so \( n<3N \). No prime occurs to odd power, so the number of
such \( n \) is less than \( \sqrt{3N} = o(N) \).

Summing, the numbers of integers less than \( N \), divisible by 12
which are \( f(n) \) for some \( n \), is at most \( N/16 + o(N) \),
while there are \( N/12 \) integers less than \( N \) divisible by 12.
So there are at least \( N/48 \) integers less than \( N \), divisible by 12,
outside the range of \( f \), and the density of the range of \( f \) is at most
\( 1 - ((1/12) - (1/16)) = 47/48 \).

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