A000931 (Padovan sequence) W. Lang, Jun 21 2010 (revisited and corrected, Oct 30 2018)

I was led to consider the Padovan sequence by a paper sent to me by A. Farina (June 11 2010):
"Expressing stochastic filters via number sequences", A. Capponi, A. Farina, C. Pilotto,

$p(n)=A000931(n+3)$, $n>=1$, is the number of partitions of the numbers $\{1,2,3,...,n\}$ into
lists of length two or three containing neighboring numbers. The 'or' is inclusive. For $n=0$ one takes $p(0)=1$.

Call the number of these lists $s_2$ and $s_3$, respectively, where $s_2$ and $s_3$ are nonnegative integers. More
precisely: $s_3$ from $\{0,1,...,\lfloor n/3\rfloor\}$ and $s_2$ from $\{0,1,...,\lfloor (n-s_3*3)/2\rfloor\}$.

The number of solutions of $n= 3*s_3 + 2*s_2$ is $A103221(n)$, $n>=0$, the number of partitions of $n$ consisting of parts 2 or 3 only.
Note that $A103221(0)=1$ from the trivial solution.

E.g., $A103221(8)=2$ from the two solutions $s_3=2$, $s_2=1$ and $s_3=0$ and $s_2=4$, corresponding to
the partitions (3,3,2) and (2,2,2,2) of 8.

Examples for the $p(n)$ combinatorics:

$p(1)=0$ because there is no solution,
p(2)=1 from $s_3=0$, $s_2=1$ and the list $[1,2]$,
p(3)=1 from $s_3=1$, $s_2=0$ and the list $[1,2,3]$,
p(4)=1 from $s_3=0$, $s_2=2$ and the lists $[1,2][3,4]$,
p(5)=2 from $s_3=1$, $s_2=1$ and the lists $[1,2,3][4,5]$ and $[1,2][3,4,5]$,
p(6)=2 from $s_3=2$, $s_2=0$ and the lists $[1,2,3][4,5,6]$ and
from $s_3=0$, $s_2=3$ and the lists $[1,2][3,4][5,6]$,
p(7)=3 from $s_3=1$, $s_2=2$ and the lists $[1,2,3][4,5][6,7]$, $[1,2][3,4,5][6,7]$, $[1,2][3,4][5,6,7]$,
p(8)=4 from $s_3=2$, $s_2=1$ and the lists $[1,2,3][4,5,6][7,8]$, $[1,2][3,4,5][6,7,8]$, $[1,2,3][4,5][6,7,8]$ and
from $s_3=0$, $s_2=4$ and the lists $[1,2][3,4][5,6][7,8]$,
p(9)=5 from $s_3=3$, $s_2=0$ and the list $[1,2,3][4,5,6][7,8,9]$ and
from $s_3=1$, $s_2=3$ and the lists $[1,2,3][4,5][6,7][8,9]$, $[1,2][3,4,5][6,7][8,9]$, $[1,2][3,4][5,6,7][8,9]$, $[1,2][3,4][5,6][7,8,9]$,
p(10)=7 from $s_3=2$, $s_2=2$ and the lists $[1,2,3][4,5,6][7,8][9,10]$, $[1,2,3][4,5][6,7,8][9,10]$, $[1,2,3][4,5][6,7][8,9,10]$, $[1,2][3,4,5][6,7][8,9,10]$, $[1,2][3,4,5][6,7][8,9,10]$, $[1,2][3,4][5,6,7][8,9,10]$ and
from $s_3=0$, $s_2=5$ and the list $[1,2][3,4][5,6][7,8][9,10]$.

etc.

Note: this is a special case of the so called (general) Morse-code polynomials. In this case only $s_3$ 3-lines (of length 2, written
in the following as a double-dash - -, for 3 neighboring points) or $s_2$ 2-lines (of length 1, written as a dash -, for 2
neighboring points) in a row of $n$ points are admitted.

Because the recurrence for $p(n)$ has no $p(n-1)$ term, there are no dots (1-lines of length 0). The classical Morse case
with only dots and 2-lines of length 1 (dashes) shows up for Fibonacci type recurrences.
E.g., the \( p(8) = 4 \) codes for \( n=8 \) are: \(-- -- --, - -- --, -- - --, \) and \(- - - - \). The numbers \( 1, \ldots, 8 \) are put at the borders of the dashes, e.g., \( 1-2-3 \) for the first double dash, or \( 4-5 \) for a second dash, etc.

Because of this combinatorial interpretation the sequence

\[
p(n) = 0^*p(n-1) + 1^*p(n-2) + 1^*p(n-3)
\]

is the fundamental sequence. As mentioned above \( p(n) = A000931(n+3) = [1, 0, 1, 1, 1, 2, 3, 4, 5, 7, 9, 12, 16, \ldots] \).

The o.g.f. \( P(x) = \sum p(n)*x^n, n=0..\infty \) = \( 1/(1-x^2-x^3) \), also showing that this is the fundamental sequence.

###

**a(a,b;n) Padovan sequences:**

The sequence \( a(n) = a(n-2) + a(n-3) \) with input \( a(-2)=a, \ a(-1)=b, \) and \( a(0)=1 \) (hence \( a(1)= a+b, \ a(2)= b+1 \)) is

\[
a(n) = a(a,b;n) = p(n) + (a+b)*p(n-1) + b*p(n-2).
\]

Therefore, \( A000931(n) = a(1,-1 ;n) = p(n) - p(n-2) = p(n-3) = [1, 0, 0, 1, 0, 1, 1, 1, 2, 2, 3, 4, \ldots] \), \( n>=0 \).

Similarly, \( A000931(n+5) = a(1,0;n) = p(n) + p(n-1) = p(n+2) = [1, 1, 1, 2, 2, 3, 4, 5, 7, 9, \ldots] \), \( n>=0 \).

also \( A007307(n+1) = a(2,0;n) = p(n) + 2*p(n-1) = p(n+2) + p(n-1) = [1, 2, 1, 3, 4, 6, 7, 10, 13, \ldots] \), \( n>=0 \).

also \( A00931(n+7) = a(1,1;n) = p(n) + 2*p(n-1) + p(n-2) = p(n+4) = [1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, \ldots] \), \( n>=0 \).

also \( A141038(n+1) = a(-1,2;n) = p(n) + 2^*p(n-1) + 2^*p(n-2) = p(n+3) + p(n-2) = [1, 1, 3, 2, 4, 5, 6, 9, 11, 15, 20, \ldots] \), \( n>=0 \).

also \( A084338(n+1) = a(0,2;n) = p(n) + 2^*(p(n-1)+p(n-2)) = p(n) + 2^*p(n+1) = p(n+3) + p(n+1) = [1, 2, 3, 3, 5, 6, 8, 11, 14, 19, 25, 33, 44, \ldots] \), \( n>=0 \).

etc.

**General input case:** \( a(a,b,c;n) Padovan sequences:**

\[
a(n) = a(n-2) + a(n-3) \text{ with input } a(-2)=a, \ a(-1)=b, \text{ and } a(0)=c \text{ (hence } a(1)= a+b, \ a(2)= b+c) \text{ is}
\]

\[
a(n) = a(a,b,c;n) = c*p(n) + (a+b)*p(n-1) + b*p(n-2),
\]
with \( p(n) := a(0,0,1;n) \).

The o.g.f. is \( P(a,b,c;x) = (c + (a+b)*x + b*x^2)/(1-x^2-x^3) \).

Therefore the Perrin sequence \( A001608(n) = a(1,-1,3;n) = 3*p(n) - p(n-2) = 2*p(n) + p(n-3) = \)
\[ 3, 0, 2, 3, 2, 5, 5, 7, 10, 12, 17, 22, 29, 39, 51, 68, 90, 119, 158, 209, 277, \ldots, \]
\( n >= 0, \)
with o.g.f. \( P(1,-1,3;x) = (3-x^2)/(1-x^2-x^3) \).

Generalized \( (A,B) \)-Padovan sequences with general input: \( a(A,B;a,b,c;n) \) (June 24, 2010)

\( a(n) = A*a(n-2) + B*a(n-3) \) with input \( a(-2)=a, a(-1)=b, \) and \( a(0)=c \) (hence \( a(1)= A*b+B*a, a(2)=A*c+B*b \)) is

\( a(n) = a(A,B;a,b,c;n) = c*p(A,B;n) + (A*b+B*a)*p(A,B;n-1) + B*b*p(A,B;n-2). \)

with \( p(A,B;n):=a(A,B;0,0,1;n) \).

The o.g.f. is \( P(A,B;a,b,c;x) = (c + (A*b+B*a)*x + B*b*x^2)/(1-A*x^2-B*x^3) \), especially \( 1/(1-A*x^2-B*x^3) \) for \( p(A,B;n) \).

Instances:

\( (2,1) \)-Padovan: \( P(2,1;a,b,c;x) = (c + (2*b + a)*x + b*x^2)/((1-x-x^2)*(1+x)). \)

\( (a,b,c)=(0,0,1): \) \( A008346(n) = \) Fibonacci(n) +(-1)^n .

\( (a,b,c)=(0,1,0): \) \( A008346(n+1), n >= 0. \)

\( (a,b,c)=(1,0,0): \) \( A008346(n-1), n >= 0, \) with Fibonacci(-1) = .1.

\( (a,b,c)=(1,0,1): \) \( A000045(n+1) = \) Fibonacci(n+1).

\( (a,b,c)=(0,1,1): \) \( A000045(n+2) = \) Fibonacci(n+2).

\( (a,b,c)=(1,1,0): \) \[ 0, 3, 1, 6, 5, 13, 16, 31, 45, 78, 121, 201, 320, 523, \ldots \].

\( (a,b,c)=(1,1,1): \) \( A066983(n+3), n >= 0. \)

\( (a,b,c)=(1,-1,1): \) \( A033999(n) = (-1)^n. \)
etc.

(1,2)-Padovan:

(a,b,c)=(0,0,1): A052947(n); (a,b,c)=(0,1,0): A052947(n+1); (a,b,c)=(1,0,0): 2*A052947(n-1).

(a,b,c)=(1,0,1): A052947(n+2); (a,b,c)=(0,1,1): A159284(n+2); n>=0.

(a,b,c)=(1,1,0): [0, 3, 2, 3, 8, 7, 14, 23, 28, 51, 74, 107, 176, 255, 390, 607, 900, 1387, 2114, 3187, 4888, ...]

(a,b,c)=(1,1,1): A159284(n+3).

(a,b,c)=(1,-1,1): A078026(n+2).

etc.

Factorization of the type (1 - A*x^2 - B*x^3) = (1 - al*x - (A-al^2)*x^2)*(1 + al*x) (June 28 2010).
Input al (alpha) and A with B = (A-al^2)*al.

Special case i)

A = 3*(al/2)^2 and B = -2*(al/2)^3 with ((1 - (1/2)*al*x)^2)*(1 + al*x)) = 1 - (3/4)*(al*x)^2 + (1/4)*(al*x)^3
with partial fraction decomposition for the o.g.f.

Pfrac(3*(al/2)^2, -2*(al/2)^3,x) :=  (3/(1 - al*x/2)^2 + 2/(1-al*x/2) + 4/(1 + al*x))/9  leading to

p((3/4)*(al)^2, -(1/4)*al^3;n) = ((3*n+5 + (-2)^(n+2))*(al/2)^n)/9.

E.g., al=2: p(3,-2;n) = A077898(n).

Special case ii)

A = 3*al^2 and B = 2*al^3 with (1 - 2*al*x)*(1 + al*x) =  1 - 3*(al*x)^2 - 2*(al*x)^3
with the partial fraction decomposition for the o.g.f.

Pfrac(3*al^2, 2*al^3,x ) :=  (4/(1-2*al*x)+ 2/(1+al*x) + 3/(1+al*x)^2)/9  leading to

p(3*al^2, 2*al^3;n) = ((3*n+5 + 2^(n+2))*al^n)/9.
E.g., \(a_l=1\): \(p(3,2;n) = A053088(n), \ n\geq 0\).

Other cases iii) \(a_l\) and \(A\) (not related like in case i) or case ii)) as input with \(B = (A-a_l^2)*a_l\).

\((1 - A*x^2 - B*x^3) = (1 - a_l*x - (A-a_l^2)*x^2)*(1 + a_l*x)\) with the partial fraction decomposition for the o.g.f.

\[
Pfrac(A, (A - a_l^2)*a_l;x) := (((A-2*a_l^2) - a_l*(A-a_l^2)*x)/(1-a_l*x-(A-a_l^2)*x^2) - (-a_l)^(n+2))/(A-3*a_l^2)
\]

leading to

\[
p(A,(A - a_l^2)*a_l;n) = ((A-2*a_l^2)*U(al,A-al^2;n) - a_l*(A-al^2)*U(al,A-al^2;n-1) - (-a_l)^(n+2))/(A-3*a_l^2)\]

with \(U(al,be;n)\) generated by the o.g.f. \(GU(al,be;x):=1/(1- al*x - be*x^2)\) ((al,be)-Fibonacci/Chebyshev).

E.g., \(a_l=1, A=2; \ B=1; (1 - 2*x^2 - x^3) = (1 - x - x^2 )* (1 + x)\); \(Pfrac(2,1;x) = x/(1-x-x^2)+1/(1+x)\); \(p(2,1;n) = F(n) + (-1)^n = A008346(n)\), with the Fibonacci numbers \(U(1,1;n-1) = F(n) = A000045(n)\).

For the explicit (Binet-de Moivre type) formula for \((A,B)\)-Padovan sequences see below.

\((A,B)\)-Padovan combinatorics (June 28 2010)

For the case \((A,B)=(1,1)\) (Padovan A000931(n+3)) see the beginning of this link.

The (generalized) Morse code uses only 3-lines of length 2, namely \(-\)\(-\)\(-\), connecting three neighboring points, and 2-lines of length 1, namely \(-\)\(-\), connecting two neighboring points. There are \(s_3\) 3-lines and \(s_2\) 2-lines, with \(s_3\) and \(s_2\) non-negative integers. If \(n\) = \(3*s_3 + 2*s_2\) then has no solution \(a(A,B;n)=0\).

Hence \(s_2 = (n - 3*s^2)/2\). Each of the \(s_3\) 3-lines receives a weight A, and each of the \(s_2\) 2-lines (dashes) a weight B. \(a(A,B;n)\) is the number of possible Morse codes of this special weighted type, namely

\[
a(A,B;n) = 0 \text{ if } n = 3*s_3 + 2*s_2\ , \text{ else}
\]

\[
\sum((1/s_3)!*((n - 2*s_3 -1*s_2)!/s_2!)*(A^s_2)*(B^s_3), s_3=0..\text{floor}(n/3)), \text{ with } s_2=s_2(n,s_3):= (n - 3*s_3)/2.
\]

E.g., \((A,B)=(2,1)\) \(a(2,1;n)= A008346(n)\) (Fibonacci(n) + (-1)^n), \(n=5\):
One solution of 5 = \(3*s_3+2*s_2\): \(s_3=1, s_2=1\) with the two codes \(-\)\(-\)\(-\)\(-\)\(-\) and \(-\)\(-\)\(-\)\(-\), weighted each with \(2^1*1^1=2\), i.e.,

\(a(2,1;5) = 2+2 = 4\).
The explicit formula for \( p(n) \) (analogon to the Binet- de Moivre formula for Fibonacci type sequences).
See also the formula for \( A000931(n) = p(n-3) \) given by Keith Schneider. Here the formula is made explicit.

\[
p(n) = \frac{r^{n+2} + c z^n + cb z^n b^n}{3r^2-1}, \quad n \geq 0,
\]

with the complex number \( c := \frac{(2r^2-1) + (r/s)I}{2} \) and its complex conjugate \( cb = \frac{(2r^2-1) - (r/s)I}{2} \), and the complex solution \( z \) to \( x^3 - x - 1 = 0 \) i.e., \( z = e + eb \), with the complex number \( e := \frac{-1 + \sqrt{3}I}{2} \) and its complex conjugate \( eb = \frac{1 + \sqrt{3}I}{2} \) (the two solutions to \( x^2 + x + 1 = 0 \)) as well as the two solutions to \( x^2 - x + 1/3^3 = 0 \), namely \( u^3 := \frac{1 + \sqrt{69}}{9}/2 \) and \( v^3 := \frac{1 - \sqrt{69}}{9}/2 \).

\( r := u + v \) and \( s := \sqrt{3}(u - v) \).

Some numbers which appear in this formula are approximately given by (10 digits, maple13):

\[
\begin{align*}
    u &: 0.9869912063, \\
v &: 0.3377267510,
\end{align*}
\]

The so called plastic number \( r \): 1.324717957, \( s \): 1.124559024, \( r/s \): 1.177988820, \( 3r^2-1 \): 4.264632998.

The complex coefficient \( c \): \(-.6623589787 + .5622795122*I\), \( c/(3r^2-1) \): \(-.1553144148+.1318471044*I\).

For the \((A,B)\)-Padovan sequences \( p(A,B;n) \), defined above, the analog explicit formulae for the case \( D(A,B) := (B^2)/4 - (A/3)^3 > 0 \) is:

\[
p(A,B;n) = \frac{r(A,B)^{n+2} + c(A,B)z(A,B)^n + cb(A,B)zb(A,B)^n}{3r(A,B)^2-A}, \quad n \geq 0,
\]

with the complex number \( c(A,B) := \frac{(2(3r(A,B)^2 - A) - A)}{6} + \frac{(A r(A,B)/(2s(A,B)))I}{2} \) and its complex conjugate \( cb(A,B) = \frac{(2(3r(A,B)^2 - A) - A)}{6} - \frac{(A r(A,B)/(2s(A,B)))I}{2} \) and the complex solution \( z \) to \( x^3 - A*x - B = 0 \). i.e., \( z(A,B) = e u(A,B) + eb v(A,B) \), with the complex number \( e := (-1 + \sqrt{3}I)/2 \) and its complex conjugate \( eb := (1 + \sqrt{3}I)/2 \) (the two solutions to \( x^2 + x + 1 = 0 \)) as well as the two solutions to \( x^2 - B*x + (A/3)^3 = 0 \), namely \( u(A,B)^3 := \frac{b}{2} + \sqrt{D(A,B)} \) and \( v(A,B)^3 := \frac{b}{2} - \sqrt{D(A,B)} \), with \( D(A,B) \) from above.

\( zb(A,B) \) is the complex conjugate of \( z(A,B) \) and
\[ r(A, B) := u(A, B) + v(A, B) \quad \text{and} \quad s(A, B) := \sqrt{3}(u(A, B) - v(A, B)). \]