ON THE NUMBER OF TOPOLOGIES DEFINABLE FOR A FINITE SET

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No general rule for determining the number \( N(n) \) of topologies definable for a finite set of cardinal \( n \) is known. In this note we relate \( N(n) \) to a function \( F_r(v_1, \cdots, v_{r+1}) \) defined below which has a simple combinatorial interpretation. This relationship seems useful for the study of \( N(n) \). In particular this can be used to calculate \( N(n) \) for small values. For \( n = 3, 4, 5, 6 \) we find \( N(3) = 29, N(4) = 355, N(5) = 7,181, N(6) = 145,807 \).

Let \( T \) be a topology on a finite set \( E \). Let \( S_1 \) be the collection of all non-empty sets in \( T \) which do not properly contain any non-empty set in \( T \). It is clear that \( S_1 \) is a collection of disjoint subsets of \( E \). If for any collection \( K \) of sets \( P \cup (K) \) denotes the set of all non-empty unions of sets in \( K \), then \( P \cup (S_1) \subseteq T \). Let \( S_1 \) be the union of all sets in \( S_1 \). Then any non-empty set in \( T \) is of the form \( U \cup V \) where \( V \in P \cup (S_1) \) and \( U \) is a subset of \( E - U \). Let \( T_1 \) be the collection of all the sets \( U \) and the null set. It can be easily proved that \( T_1 \) is a topology on \( E - S_1 \). We shall refer to \( S_1 \) and \( T_1 \) as "nucleus" and "orbital topology" of the topology \( T \), respectively.

By a "reduced base" of a topology on a finite set we shall mean a base such that no base set is a union of other base sets.

**Theorem.** Let \( B \) be a reduced base for \( T \). Then there is a unique single-valued mapping \( f : B \to P \cup (S_1) \) such that \( B = \{ X_1 \cup X_1/f, X_1 \in B_1 \} \cup S_1 \) is a reduced base for \( T \). Also, \( f \) preserves the inclusion relation \( \subseteq \) for sets. Conversely if \( S_1 \) is a non-empty collection of disjoint non-empty subsets of \( E \), \( T_1 \) is any topology on \( E - S_1 \) and \( f \) is a single-valued mapping from a reduced base \( B \) for \( T_1 \) into \( P \cup (S_1) \) which preserves \( \subseteq \) then \( B = \{ X_1 \cup X_1/f, X_1 \in B_1 \} \cup S_1 \) is a reduced base for a topology \( T \) on \( E \) such that \( S_1, T_1 \) are respectively the nucleus and the orbital topology of \( T \).

**Proof.** For any \( X_1 \in B_1 \), we define \( X_1/f \) to be a member of \( P \cup (S_1) \) such that \( X_1 \cup X_1/f \in T \) and \( X_1 \cup V \notin T \) if \( X_1/f \not\subseteq V \). \( X_1/f \) exists because \( T_1 \) is the orbital topology of \( T \). If \( V \in P \cup (S_1) \) has the property that \( X_1/f \) then \( V \supseteq X_1/f \) and \( X_1/f \subseteq V \), so that \( X_1/f = V \). Thus \( f \) is a mapping from \( B \) into \( P \cup (S_1) \). We show that \( f \) is the mapping required by the first
part of the theorem. Let \( X_1 \subseteq X_1' \); then
\[
(X_1 \cup X_1f) \cap (X_1' \cup X_1'f) = X_1 \cup (X_1f \cap X_1'f) \in T,
\]
since \( X_1, X_1' \) are disjoint for all \( X_1, X_1' \in B_1 \). We conclude from the definition of \( f \) that \( X_1 \cap X_1' = X_1f \cap X_1'f \) so that \( X_1f \subseteq X_1' \) and hence \( f \) preserves \( \subseteq \). Next let \( Y \in T \) and let \( Y = U \cup V \), where \( U \in T_1, V \in P_\cup (S_1) \). Since \( B_1 \) is a base for \( T_1 \) we can write \( U = U' \cup B_1' \) for some subcollection \( B_1' \) of \( B_1 \). If \( U \) is empty, \( Y \) is trivially a union of sets in
\[
B = \{ X_1 \cup X_1f, X_1 \in B_1 \} \cup S_1.
\]
Hence we can suppose \( B_1' \) non-empty. Then \( X_1' \subseteq V \) for every \( X_1' \in B_1' \); for
\[
X_1' \cup (V \cap X_1f) = (U \cup V) \cap (X_1' \cup X_1'f) \in T
\]
and therefore \( V \cap X_1f = X_1'f \). Hence \( Y = \cup \{ X_1' \cup X_1'f, X_1' \in B_1' \} \cup (\text{union of sets in } S_1) \). This proves that \( B \) is a base for \( T \). That \( B \) is reduced follows directly from the definition of \( f \) and the assumption that \( B_1 \) is reduced. To prove the uniqueness of the mapping \( f \) suppose that \( f^* \) is another mapping satisfying the first part of the theorem. Then, for some \( X_1 \in B_1, X_1f \subseteq X_1'f \). But \( X_1 \cup X_1f \in T \) and therefore is a union of sets in
\[
B^* = \{ Y_1 \cup Y_1f^*, Y_1 \in B_1' \} \cup S_1.
\]
Since \( B_1 \) is reduced is impossible in view of \( X_1f \subseteq X_1'f \).

For the converse, let \( B \) be as defined in the theorem. Then \( E = \cup B = (\cup B_1) \cup (\cup S_1) \). Let \( Y, Y^* \) be any two members of \( B \) and write \( Y = X_1 \cup X_1f, Y^* = X_1^* \cup X_1^*f \). Since \( f \) preserves \( \subseteq \),
\[
Y \cap Y^* = (X_1 \cap X_1^*) \cup (X_1f \cap X_1^*f)
= (X_1 \cap X_1^*) \cup (X_1 \cap X_1^*)f \cup (\text{union of sets in } S_1).
\]
Now \( X_1, X_1^* \in B_1 \) and \( X_1 \cap X_1^* = \emptyset \cup B_1' \), where \( B_1' \) is a subcollection of \( B_1 \). Since \( Z_1f \subseteq (X_1 \cap X_1^*)f \) for every \( Z_1 \in B_1' \), this gives
\[
Y \cap Y^* = \cup \{ Z_1 \cup Z_1f, Z_1 \in B_1' \} \cup (\text{union of members of } S_1);
\]
so that \( Y \cap Y^* \) is a union of members of \( B \). In case one or both of \( Y, Y^* \) are members of \( S_1 \) and therefore not expressible in the form \( X \cup Xf \), \( Y \cap Y^* \) is trivially a union of sets in \( B \). Hence the intersection of any two members of \( B \) is a union of members of \( B \) and therefore \( B \) is a base for a topology \( T \) on \( E \). The rest of the theorem now follows directly.

For any topology \( T \) on a finite set \( E \) we can form the sequence \( T_0 = T, (S_1, T_1), (S_2, T_2), \ldots, (S_t, T_t), \ldots, (S_{t+1}, T_{t+1}) \), where \( S_z, T_z \) are respectively the nucleus and the orbital topology of \( T_{k-1} \) for \( t \geq k \geq 1 \) and \( S_{t+1} \) is a reduced base as well as the nucleus of \( T_t \), so that \( T_t = P_\cup (S_{t+1}) \). By the above theorem there is a unique sequence of mappings \( f_1, \ldots, f_t \) such that for
1 \leq i \leq t, f_i maps B_i into \bigcup_i S_i, where B_i is a reduced base for T_i and is defined by

\[ B_i = S_{t+1} \cup B_i = \{X_{t+1} \cup X_{i+1} f_{i+1}, X_{i+1} \in B_{i+1}\} \cup S_{t+1}, \]

for \( 0 \leq i \leq t \).

By our theorem, every topology on E can be obtained as follows:

- Partition \( E \) into any number, say \( r \), of disjoint and collectively exhaustive classes \( E_1, \ldots, E_r \) and then partition, in an arbitrary way, the set \( \{E_1, \ldots, E_r\} \) into disjoint and collectively exhaustive classes, say, \( S_1, \ldots, S_t \).
- Let \( f_1, \ldots, f_t \) be any mappings such that
  1. \( f_1 \) maps \( B_1 = S_{t+1} \) into \( \bigcup_i S_i \).
  2. \( f_{i-1} \) maps \( B_{i-1} \) into \( \bigcup_i S_{i-1} \) where
     \[ B_{i-1} = \{X \cup X f_{i-1}, X \in B_{i-1}\} \cup S_{i-1}. \]
  3. Each of the mappings \( f_1, \ldots, f_t \) preserves the inclusion relation \( \subseteq \) for sets.

Then \( B = B_0 = \{X_1 \cup X_1 f_1, X_1 \in B_1\} \cup S_1 \) is a base for a topology on \( E \) and every topology on \( E \) is obtained in this way.

In view of this we can express the number \( N(n) \) of topologies definable for a finite set of cardinality \( n \) as follows:

\[ (1) \quad N(n) = \sum_{r=1}^{n} \left[ M_{n,r} r! \sum_{r_1+\cdots+r_t=r} \left( \frac{|F_t(r_1, \ldots, r_t)|}{r_1! \cdots r_t!} \right) \right] \]

where \( M_{n,r} \) is the number of ways a set of order \( n \) can be partitioned into \( r \) unordered classes and \( F_t(r_1, \ldots, r_t) \) is the number of sequences of mappings \( f_1, \ldots, f_t \) described above when \( S_1, \ldots, S_t \) have \( r_1, \ldots, r_t \) members respectively. The summation in curly brackets extends over all finite sequences \( r_1, \ldots, r_t \) of positive integers satisfying \( r_1 + \cdots + r_t = r \).

The following recurrence relation holds for \( M_{n,r} \):

\[ M_{n+1,r} = rM_{n,r} + M_{n,r-1}. \]

The function \( F_t(r_1, \ldots, r_t) \) has a simple combinatorial interpretation which we explain by taking \( t = 3 \) and by referring to the figure below.

| \( x(1,1) \) | \( x(1,1) \) |
| \( x(2,1) \) | \( x(2,1) \) |
| \( x(3,1) \) | \( x(3,1) \) |
| \( x(4,1) \) | \( x(4,1) \) |

Figure 1

[1] Strictly speaking, these formulae are represented by the following conclusions about \( t \):

\[ (3) \quad F_t(r_1, \ldots, r_t) = \frac{M_{n,r} r! \sum_{r_1+\cdots+r_t=r} \left( \frac{|F_t(r_1, \ldots, r_t)|}{r_1! \cdots r_t!} \right)}{n!} \]

\[ (4) \quad F_1(r_1, r_2) = \frac{M_{n,r} r! \sum_{r_1+\cdots+r_t=r} \left( \frac{|F_t(r_1, \ldots, r_t)|}{r_1! \cdots r_t!} \right)}{n!} \]

\[ (5) \quad F_2(r_1, 1, k) = \frac{M_{n,r} r! \sum_{r_1+\cdots+r_t=r} \left( \frac{|F_t(r_1, \ldots, r_t)|}{r_1! \cdots r_t!} \right)}{n!} \]

\[ (6) \quad F_3(1, 1, \ldots, 1) = \frac{M_{n,r} r! \sum_{r_1+\cdots+r_t=r} \left( \frac{|F_t(r_1, \ldots, r_t)|}{r_1! \cdots r_t!} \right)}{n!} \]

\[ (7) \quad F_4(1, 1, \ldots, 1) = \frac{M_{n,r} r! \sum_{r_1+\cdots+r_t=r} \left( \frac{|F_t(r_1, \ldots, r_t)|}{r_1! \cdots r_t!} \right)}{n!} \]
In this figure we have taken $e_1 = r_4$, $e_2 = r_3 + r_4$, $e_3 = r_2 + r_3 + r_4$, $e_4 = r_1 + r_2 + r_3 + r_4$. Every one of the $r_4$ squares in the first row is given to be occupied with just one of the symbols $x(1, 1), \ldots, x(1, e_1)$ that are labels for sets in $S_4$. In the second row only the last $r_2$ squares on the right are given to be initially occupied, each by just one of the $r_2$ symbols $x(2, e_1 + 1), \ldots, x(2, e_4)$ that similarly stand for sets in $S_2$; and so on. Let us refer to the $j$th square from the left in the $i$th row from the top as $a(i, j)$. In what follows we shall not explicitly mention the restrictions on the ranges of the variables $i, j, k, \ldots$. Write $\Sigma(i, j) = \{x(i, j)\}$ if $a(i, j)$ is not initially empty. The combinatorial problem now is to place in every empty square $a(i, j)$ a non-empty set $\Sigma(i, j)$ of symbols such that

(iv) $\Sigma(i, j) \subseteq \{x(i, e_{i-1} + 1), \ldots, x(i, e_i)\}$,

(v) $x(i, k) \in \Sigma(i, j)$ implies $\Sigma(i + 1, k) \subseteq \Sigma(i + 1, j)$.

Thus, for example, the conditions (iv), (v) compel us to place in the empty squares of the third row in Fig. 1 symbols chosen from $x(3, e_2 + 1), \ldots, x(3, e_3)$, and if $x(3, e_2)$ has been placed in $a(3, e_2)$ (the square immediately below the one containing $x(2, e_2)$) then $x(3, e_3)$ will have to occur in any set of symbols to be placed in a square of the third row which comes directly under a square containing $x(2, e_2)$. Let $Y(i, k) = \bigcup_{l=1}^{k} \Sigma(l, k)$. Then it is easily seen that if we let $B_{4-i}$ be the set of all $Y(i, k)$ for fixed $i$ and write $Y(i, k)/a_{4-i} = \Sigma(i + 1, k)$ then $B_{4-i}, I_{4-i}$ satisfy (i), (ii), (iii) for $t = 3$. It follows that $F_4(r_1, r_2, r_3, r_4)$ is the number of ways of placing the symbols $x(i, j)$ in the empty squares of Fig. 1 such that (iv) and (v) are satisfied.

We can use this interpretation of $F_4(r_1, \ldots, r_{t+1})$ to prove the following formulae.

(3) $F(r_1) = 1$,

(4) $F_1(r_1, r_2) = (2^{r_1} - 1)^{r_2}$,

(5) $F_2(r_1, 1, r_3) = \sum_{l=1}^{r_1} \binom{r_1}{l} 2^{(r_2-1) r_3}$,

(6) $F_2(1, r_1, r_2) = \sum_{l=1}^{r_1} \sum_{m=0}^{r_2} 2^{r_3-m} \binom{r_1}{l} \binom{r_2}{m} (2^{r_2-1} r_1-1)^{(2^m - 1) - m(2^m - 1)}$,

(7) $F_4(1, 1, \ldots, 1, r_{i+1}) = \sum_{j_1 > 0, j_2 > 0, \ldots, j_i > 0} \binom{r_{i+1}}{j_1} \binom{r_{i+1}-1}{j_2} \ldots \binom{r_{i+1} - (j_1 + \cdots + j_{i-1})}{j_i}$.

1 Strictly speaking, members of $B_{4-i}$ must be taken as the unions $\bigcup Y(i, k)$ of all sets represented by the $x$'s in $Y(i, k)$, but since $x$'s represent disjoint sets this will not affect our conclusion about $F_4(r_1, \ldots, r_4)$. 

\section*{Topologies definable for a finite set}

197
As an illustration we prove (5). We have to consider the number of ways some of the \(x(i,j)\) can be placed in the empty squares in fig. 2 below such that (iv), (v) are satisfied.

![Figure 2]

In every empty square of the second row of this figure we must put just \(x(2, e_3+1)\). In the square \(\sigma(3, e_3+1)\) under \(x(2, e_3+1)\) we can place any subset \(\Sigma(3, e_3+1)\) of \(\{x(3, e_3+1), \ldots, x(3, e_4)\}\). In the remaining empty squares of the third row we must put every symbol in \(\Sigma(3, e_3+1)\) in addition to some other symbols arbitrarily selected from \(\{x(3, e_3+1), \ldots, (3, e_3)\} - \Sigma(3, e_3+1)\).

The formula (5) is now obvious.

We have employed formulae (1) – (7) in calculating \(N(n)\) for \(n = 3, 4, 5, 6\).

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