AN APPLICATION OF GENERATING SERIES
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Although the notion of generating series is a very important one in number theory, combinatorial analysis, probability theory and other branches of mathematics, there are perhaps not enough well-known examples which illustrate the applicability of this notion in different situations. In this note we present what is probably a new example of this type. The results we derive could be obtained by purely elementary methods as well, but this does not detract from the elegance of the generating series approach.

Let us seek a sequence of non-negative integers \( A = \{a_1 < a_2 < \ldots \} \) such that every non-negative integer \( n \) can be represented uniquely in the form \( n = a_i + 2a_j \). To this end we define a function \( f(x) \) by

\[
 f(x) = \sum_{i=1}^{\infty} x^{a_i}, \quad |x| < 1.
\]

Now the number of representations of each non-negative integer \( n \) in the form \( a_i + 2a_j \) will be the coefficient of \( x^n \) in the expansion of \( f(x)f(x^2) \). Thus, since each integer is to have a unique representation we must have

\[
 f(x)f(x^2) = 1 + x + x^2 + x^5 + \ldots = \frac{1}{1-x}.
\]

From (2) we obtain

\[
 \frac{f(x)f(x^2)}{f(x^2)f(x^4)} = \frac{1-x^2}{1-x}
\]
or

\[
 f(x) = (1+x)f(x^4).
\]

Iteration of (4) and taking into account the fact that \( f(x^n) \rightarrow 1 \) as \( n \rightarrow \infty \) yields

\[
 f(x) = (1+x)(1+x^4)(1+x^{16})(1+x^{64}) \ldots.
\]

The coefficient of \( x^n \) on the right hand side of (5) is the number of representations of \( n \) as the sum of distinct powers of 4. Since every integer has a unique representation as a sum of distinct powers of 2, a number will have at most one representation as a sum of distinct powers of 4 and our argument shows that the set \( A \) must consist precisely of those numbers which have such a representation. The argument actually shows that if a set \( A \) exists and has the required properties then it must be the unique set described above, but having found the set it is easily seen that it does indeed have the required properties. The required set begins with

\[\{0, 1, 4, 5, 16, 17, 20, 21, 64, \ldots \}.\]
A by-product of this argument is still another example of an explicit 
(1 - 1) correspondence between the non-negative integers \( n \) and the posi-
tive pairs of integers \((i, j)\). This is an immediate consequence of the unique 
 solvability of the equation \( n = a_i + 2a_j \), i.e. given \( n \) the \( i \) and \( j \) are uniquely 
determined.

As a further application of the method one can prove, in similar fash-
ion, that for every integer \( k > 1 \) there is a unique set \( A_k \) of non-negative 
integers such that every integer \( n \) can be uniquely expressed in the form 
\( n = a + bk \), with \( a \) and \( b \) elements of \( A_k \). On the other hand such a set can-
not exist for \( k = 1 \) for then with \((x)\) defined as in (1),

\[
(6) \quad f^2(x) = \frac{1}{1-x}
\]

so that

\[
(7) \quad f(x) = (1-x)^{-16}
\]

which does not have an expansion of the form (1).

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**RATIONAL APPROXIMATIONS OF \( e \)**

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It is well-known that \((2721)/(1001) \approx 2.7182817\) approximates the 
value of \( e \), being accurate to 6 decimal places. This is equivalent to

\[
e = \frac{4}{7} + \frac{16}{11} + \frac{9}{13} = \frac{11}{7} + \frac{5}{11} + \frac{9}{13} = \frac{4}{7} + \frac{5}{11} + \frac{22}{13}.
\]

In all three cases, the denominators are consecutive primes. In the first 
sum, the numerators are consecutive squares. In the second sum, the num-
erators are all odd and their sum is the next consecutive square. In the 
third sum, the sum of the numerators equals the sum of the denominators, 
i.e. 31.

The approximating fraction may also be written as

\[
\frac{877 + 907 + 937}{7 \cdot 11 \cdot 13}
\]

in which the denominator is the product of three consecutive primes and 
the numerator is the sum of three primes in arithmetic progression. The 
numerator may be written as the sum of three primes in A. P. in 25 other 
ways, in all of which 907 is the mean. The smaller terms of the A. P.'s

(Continued on page 54.)

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**TEACHING**

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**THE DISTANCE FORMULA**

THOMAS E. MOTT,

As a mathematician, I ha-

is a very orderly and consis-
t task of converting others to the
ful or useful is to convert one in
consistencies of this nature. The
sign of this distance. There are
three such conventions, two of
subject.

Since there are a number of
la in Analytic Geometry, we
must first of all consider the
value of the point. However, a proof
as to be found in "Analytic Geo-
the most suitable; for I do not
\( ax + by + c = 0 \) is the equation of

\[
d = \sqrt{ax^2 + by^2 + cz^2}
\]

is the distance from this line
that we have yet to adopt a con-
of distance from the line to the
with oblique lines, hence \( a \cdot b \).

The first convention under

can be the same as the sign of
is positive when \( p \) is "above"
the line. But what is meant by
"above"? For \( p \) "above the
vertical projection of \( p \) on the
in the \( y \) sense" means that \( t \)
above \( p \). The proof of this propo-

The line \( ax + by + c = 0 \) di-
half plane "above" the line and
half planes contain respec-

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