Ibn al-Haytham's way to prove a recurrence for the sums of the k-th power of the first positive integers \( S(k,n) := \sum(j^k, j=1..n) \).

See the H. K. Strick reference given in A000537, p. 13. Here written for the fourth power \( k=4 \), but one can take any power \( k \geq 1 \) instead.

Start with

\[
\sum(j^3, j=1..n) \times (n + 1) = \\
(1^3 + 2^3 + \ldots + n^3) (n + 1) = \\
1^3 \times (1 + 1 + 1 + \ldots + 1) \quad \text{[there are n+1 1s in the bracket]} \\
+ 2^3 \times (2 + 1 + 1 + \ldots + 1) \quad \text{[there are n-1 1s in the bracket]} \\
+ 3^3 \times (3 + 1 + \ldots + 1) \quad \text{[there are n-2 1s in the bracket]} \\
\ldots \\
+ (n-1)^3 \times ((n-1) + 1 + 1) \\
+ n^3 \times (n + 1)
\]

Now sum the diagonals. In the first diagonal put \( 1^3 \times 1 = 1^4 \) in the others put \( 1^3 \times 1 = 1^3 \).

\[
= 1^3 \times 1 + 2^3 \times 2 + 3^3 \times 3 + \ldots + n^3 \times n \quad \text{[sum(j^4, j=1..n) = S(4, n)]} \\
+ 1^3 \times 1 + 2^3 \times 1 + 3^3 \times 1 + \ldots + n^3 \times 1 \quad \text{[sum(j^3, j=1..n) = S(3, n)]} \\
+ 1^3 \times 1 + 2^3 \times 1 + \ldots + (n-1)^3 \times 1 \quad \text{[sum(j^3, j=1..n-1)=S(3,n-1)]} \\
\ldots \\
+ 1^3 \times 1 + 2^3 \times 1 \quad \text{[sum(j^3, j=1..2) = S(3, 2)]} \\
+ 1^3 \times 1 \quad \text{[sum(j^3, j=1..1) = S(3, 1)]}
\]

\[
= S(4,n) + \sum(S(3,m), m=1..n).
\]

Therefore: \( S(4,n) = S(3,n) \times (n+1) - \sum(S(3,m), m=1..n), \ n \geq 1 \).

One can cancel the \( S(3,n) \) terms:

\[
S(4,n) = S(3,n) \times n - \sum(S(3,m), m=1..n-1), \ n \geq 1,
\]

where the undefined sum for \( n=1 \) is put to 0.

As mentioned above, one can take any power \( k \geq 1 \) in this calculation, thus

\[
S(k,n) = S(k-1,n) \times (n+1) - \sum(S(k-1,m), m=1..n) \\
= S(k-1,n) \times n - \sum(S(k-1,m), m=1..n-1), \ n \geq 1, \ k \geq 1.
\]

-------------------- e.o.f. -------------------------