July 27, 1970

Neil:

You are right about Comtet's table. I enclose my table of the number of (fully) labeled essentially series series-parallel networks: $n = 1(1)20$

The connection of the numbers $Z_n \equiv Z_n(1,1,\ldots 1)$ - Comb. Identities, p. 197 with Schröders fourth problem is as follows

First

$$Z_n/(n+1)! = B_n\left(\tfrac{1}{2!}, \ldots, \tfrac{1}{(n+1)!}\right) \equiv B_n(A_1, \ldots, A_n), \quad A_j = \tfrac{1}{(j+1)!}$$

and C.I. p. 149

$$y = x - A_1x^2 - A_2x^3 - \ldots$$
$$x = y + B_1y^2 + B_2y^3 + \ldots$$

Hence

$$y = x - \frac{x^2}{2!} - \frac{x^3}{3!} - \ldots = 2x - (e^x - 1) = 1 + 2x - e^x$$
$$x = y + \frac{Z_1 y^2}{2!} + \frac{Z_2 y^3}{3!} + \ldots = \sum_1^\infty s_n \frac{y^n}{n!} \quad s_0 = s_1 = 1 \quad s_n = Z_{n-1}$$

The equation

$$e^x - 1 - 2x + y = 0$$

is the solution of Schröder's fourth problem

I haven't checked Riordan & Shannon, The number of two-terminal series-parallel networks, Journal of Math & Phy. 21 (1942), 83-93 on asymptotic results, but I note that R. M. Foster in Math Reviews 4 (1943) p. 951 says "no true asymptotic formulae are found".

Yours

John

P.S. The table in R&S goes to 30

311

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Numbers of labeled essentially series series-parallel networks

$Y_n(a_0 \ldots a_n) = 2a_n \quad n = 2, 3, 4, \ldots \quad a_1 = 1$

(Congruences to prime moduli: $a_p \equiv a_{p-1} \equiv 1\ (p)$ $\quad a_{p+1} \equiv 2\ (p)$ $\quad a_{p+k} \equiv 2a_{k+1}\ (p)$ $\quad a_{(p-1)k} \equiv 2^k a_k\ (p)$

n			a_n		(311)
1					1
2					1
3					4
4					26
5					236
6					2752
7					39208
8				6	60032
9				128	18912
10				2821	37824
11				69398	97856
12			18	86661	82784
13			561	73490	20544
14			18179	07032	09728
15		6	35372	60424	86272
16		238	51397	09652	57728
17		9571	02058	64190	12608
18	4	08837	90566	04440	10496
19	185	22305	41036	49869	06624
20	8870	94711	30411	93473	88416

JR

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311