Nov 10, 1975

Dear Neil

I have finally awakened to the fact that if $g(t) = t\,a(t)$, $G(t) = t\,A(t)$ and $g(G(t)) = G(g(t)) = t$ then, if $a(0) = A(0) = 1$,

$$a_n = B_n(-A_1 \quad , -A_n)$$
$$A_n = B_n(-a_1 \quad , -a_n)$$

with

$$B_n(b_1, \quad , b_n) = \frac{1}{n+1} \sum \binom{n+k}{n} \frac{k!}{k_1! \cdots k_n!} b_1^{k_1} \cdots b_n^{k_n}$$

the reversion of series multivariable polynomial (Comb. Id. p. 149)

Here are some consequences for the Handbook.

1. $B_n(1,1,0 \ldots 0)$ is sequence 1146. But

$$B_n(1,1, \ldots 0) = \frac{1}{n+1} \sum_0 \binom{n-j}{j}\binom{2n-j}{n-j} = \frac{1}{n+1} f_n^{(n+1)}$$

$f_n^{(k)}$ the kth convolved Fibonacci C. I, p. 89

2. $B_n(1,1, \ldots 1) = H_n(2)$, with $H_n(z)$ John Morrison's polynomial (C I, 151), which is sequence 1163. These also the numbers of Schröder's third problem (as already noticed in your references). Incidentally, if $g(t) = t\,H(t) = \sum_0 H_n(1) t^{n+1}$, then $G(t) = t(1-2t)(1-t)^{-1}$ and $G(g(t)) = t \Rightarrow (1+t)H(t) - 2t\,H^2(t) = 1$, agreeing with Motzkin (and Schröder)

3. Somehow or other I started a reexamination of $c^{\#}(x)$; $C(x) = \sum c_n x^n$ $c_n = \frac{1}{n+1}\binom{2n}{n}$ - Catalan. The enclosed table is the

result. Notice that the following identifications are made

j	1	3	4	5	7	9	11	13
Handbook	577	1130	1415	1682	1866	1981	2047	2104

Of course (C. I. p. 153) $c_n(j) = \frac{j}{2n+j}\binom{2n+j}{n} = \frac{j}{n}\binom{2n-1+j}{n-1}$

Notation: $C^j(x) = \sum c_n(j)\, x^n$

My week was made last Monday by a letter from Carleson that Acta Math will publish (next year, of course) my paper "The Blossoming of Schröder's Fourth Problem"

As always, best to you and love to Ann

John

Convolution Catalan numbers

$c^r(x) = \sum C_n(r) x^n$

$C_n(r) = C_n(r+1) + C_{n-1}(r+2) = \sum (-1)^k \binom{n-1-k}{k} C_{n+r+k} = \binom{2n+r}{n} - \binom{2n+r}{n-2} = \frac{r}{n}\binom{2n+r-1}{n-1}$

Handbook	n	0	1	2	3	4	5	6	7	8	9	10
577	1	1	1	2	5	14	42	132	429	1430	4862	16796
"	2	1	2	5	14	42	132	429	1430	4862	16796	58786
1136	3	1	3	9	28	90	297	1001	3432	11934	41990	149226
1415	4	1	4	14	48	165	572	2002	7072	25194	90440	326876
1662	5	1	5	20	75	275	1001	3640	13260	48450	177650	653752
Absent	6	1	6	27	110	429	1638	6188	23256	87210	326876	1225785
1666	7	1	7	35	154	637	2548	9996	38760	149226	572033	2187185
Absent	8	1	8	44	208	910	3808	15504	62016	245157	961400	3749460
1991	9	1	9	54	273	1260	5508	23256	95931	389367	1562275	6216210
—	10	1	10	65	350	1700	7752	33915	144210	600875	2466750	10015005
2057	11	1	11	77	440	2244	10659	48279	211508	904475	3798795	15737865
Absent	12	1	12	90	544	2907	14364	67298	303595	1332045	5720520	24192090
2054	13	1	13	104	663	3705	19019	[illegible]	[illegible]	1924065	8454225	36419[illegible]
	14	1	14	119	798							
	15	1	[illegible]	[illegible]	950							
	-1	1	-1	-1	-2	-5	-14	-42	-132	-429	-1430	-4862
	-2	1	-2	-1	-2	-5	-14	-42	-132	-429	-1430	-4862
	-3	1	-3	0	-1	-3	-9	-28	-90	-297	-1001	-3432
	-4	1	-4	2	0	-1	-4	-14	-48	-165	-572	-2002
	-5	1	-5	5	0	0	-1	-5	-20	-75	-275	-1001
	-6	1	-6	9	-2	0	0	-1	-6	-27	-110	-429
	-7	1	-7	14	-7	0	0	0	-1	-7	-35	-154
	-8	1	-8	20	-16	2	0	0	0	-1	-8	-44
	-9	1	-9	27	-32	9	0	0	0	0	-1	-9
					A.13	CT 553	$C_n(r)$	$= \frac{r}{2n+r}\binom{2n+r}{n}$		$+ \frac{r}{n}\binom{2n+r-1}{n-1}$		

GN-553-2 (12-[illegible])

[illegible]

669
1628
1190
1859

GN-553 (12-55)

1628

Planted trees with no pts of degree 2 & root ignored
by number of pts & no of endpoints

$(1+x)\,Q(x,y) + x - xy = x \exp\left[\, q(x,y) + \frac{q(x^2,y^2)}{2} + \cdots \right]$ $\qquad Q(x,y) = \sum x^n Q_n(y) = \sum x^n y^m$

$Q_{n,n-1} = 1$	Row Sum	Column Sum m \ n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
Q_{n}			1	0	1	1	2	3	6	10	19	35	67	127	248	462	952	1885	3765	7546	15221	30802	$Q_{n,m}$
$Q_{n,n-2} = \binom{n-4}{1}$	1	1	1	0																			
	1	2			1																		
$Q_{2n-1,n} = b_n = 1190$	2	3				1	1																
= no. of p... polynom.	5	4					1	2	2														
	12	5						1	3	5	3												
Cur... I, 68	33	6							1	4	10	12	6										
	90	7								1	5	16	29	28	11								
	261	8									1	6	24	57	84	66	23						
	766	9										1	7	33	99	192	231	157	46				
Br (p.203)	2312	10											1	8	44	157	382	615	634	373	98		207
	7068	11												1	9	56	234	682	1380	1905	1704	890	2
	21965	12													1	10	70	333	1133	2755	4782	5746	
not planted?	68954	13														1	11	85	456	1772	5043	10536	
	218751	14															1	12	102	606	2652	8635	
	699534	15																1	13	120	786	3819	
																			1	14	140	998	
669																				1	15	161	1275 12
1859 → $Q_{n,n+3}$			2	5	10	16	24	33	44	56	70	85	102	120	140	161	184	208	234 261	290 320	352 385	16	189
type I			3	5	6	8	9	11	12	14	15	17	18	20	21	23	24	26	27 29	30 32	33	1	17
			2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2 1	2 1			1
1860 $Q_{n,n-4}$			3	12	29	57	99	157	234	333	456	106	786	998	1245		1 2	3 4	5 6 7 8	9 10 11	12 13	14 15	16
type I			9	17	28	42	58	77	99	123	150	180	212	247		b_n	1 1	1 2	3 6 11 23	46 98 207	451 983	2179 4850	10905
			8	11	14	16	19	22	24	27	30	32	35				1190						
			3	3	2	3	3	3	3	3	2	3	3	2									

VK 10/14/70

66
162
119
185

GN-553 (12-55)

1628

Planted trees with no pts of degree 2 + root ignored
by number of pts + no of endpoints

$(1+x)Q(x,y) + x - xy = x \exp\left[Q(x,y) + \frac{Q(x^2,y^2)}{2} + \dots\right]$

$Q(x,y) = \sum x^n Q_n(y) = \sum x$

$Q_{n,n-1}=1$	Row Sum	Column Sum / m \ n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
		$Q_{n,m}$	1	0	1	1	2	3	6	10	19	35	67	127	248	462	952	1885	3765	7546	15221	30802
$Q_{n,n-2} = \binom{n-4}{1}$	1	1	1	0																		
	1	2			1																	
$Q_{2n-1,n} = b_n$ =1190	2	3				1	1															
= no. of planted trees	5	4					1	2	2													
random.	12	5						1	3	5	3											
Cf. I, 68	33	6							1	4	10	12	6									
	90	7								1	5	16	29	28	11							
	261	8									1	6	24	57	84	66	23					
(p.303)	766	9										1	7	33	99	192	231	157	46			
Bc	2312	10											1	8	44	157	382	615	634	373	98	
	7068	11												1	9	56	234	682	1380	1905	1704	890
	21965	12													1	10	70	333	1133	2755	4782	5746
	68954	13														1	11	85	456	1772	5043	10536
	218751	14															1	12	102	606	2652	8635
	699634	15																1	13	120	786	3819
																			1	14	140	998
669																				1	15	161
1859 → $Q_{n,n-3}$			2	5	10	16	24	33	44	56	70	85	102	120	140	161	184	208	234 261	290 320	352 385 1	16
type 1			3	5	6	8	9	11	12	14	15	17	18	20	21	23	24	26	27 29	30 32	33.	1
			2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2 1	2 1		
1860 $Q_{n,n-4}$			3	12	29	57	99	157	234	333	458	606	786	998	1245	n	1 2	3 4	5 6 7 8	9 10 11	12 13	14 15
type 1			9	17	28	42	58	79	99	123	150	180	212	247		b_n	1 1	1 2	3 6 11 23	46 98 207	451 983	2179 4850
			8	11	14	16	19	22	24	27	30	32	35				1190			1/2		