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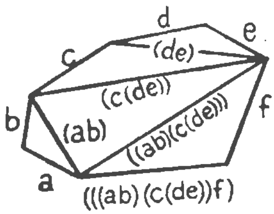
SOME PROBLEMS IN COMBINATORICS

By H. G. Forder

The following problems are merely different forms of one:

1. In how many ways can a convex polygon of $n+1$ sides be split into triangles by non-intersecting diagonals? We represent such a splitting by a symbol. Take one side of the polygon as base and letter the other n sides in order by $a, b, c, \ldots$. If a triangle with sides lettered x, y is cut off, denote the diagonal by (xy). If this diagonal and side z form the sides of another triangle of the dissection, denote the third side of that triangle by $((xy)z)$. Similarly for two diagonals, as the figure illustrates. Finally the symbol for the unlettered base is the symbol for the dissection.



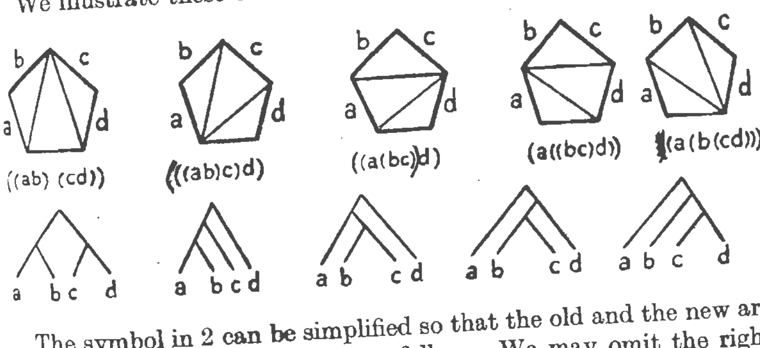
Such a symbol then has the following form. We have n letters in a *fixed* order; any two adjacent letters can be combined, then any two adjacent combinations, or a combination and an adjacent letter, to left or right, can be combined together. Thus our problem is:

2. How many "binary associations" can be formed from n letters in fixed positions?

These binary associations can be represented by trees.

3. How many trees can be formed with n ends, each fork having two branches?

We illustrate these transformations by a simple case.



The symbol in 2 can be simplified so that the old and the new are in one-to-one correspondence, as follows. We may omit the right

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hand brackets writing

$$((ab(cd \quad (((abcd \quad ((a(bcd \quad (a((bcd \quad (a(b(cd$$

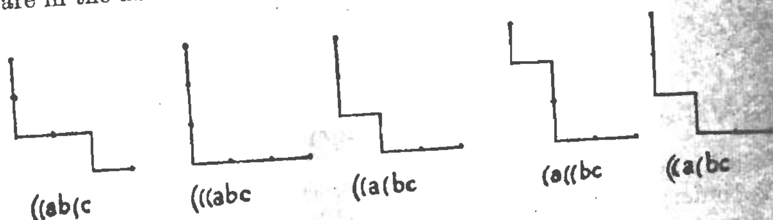
This reminds one of the Polish notation in symbolic logic. Furthermore we can omit the last letter, getting

$$((ab(c \quad (((abc \quad ((a(bc \quad (a((bc \quad (a(b(c$$

This abridged form can be completed in just one way to make the earlier form.

It will be seen that now the number of brackets equals the number of letters and that at any stage in traversing the symbol from left to right the number of brackets is never less than the number of letters.

Now take a chess board. Start from the top left square and interpret the symbol as follows: a bracket means, move down one square; a letter means move one square to the right. Thus our moves are in the above cases:



Our problem has now been transformed to:

4. How many paths of this kind starting at the top-left corner end on the diagonal?

If we note that each square can be reached either from the square just above it or from the square on its left, we can construct the following table for the number of paths to any square below the diagonal.

1					
1	1				
1	2	2			
1	3	5	5		
1	4	9	14	14	
1	5	14	28	42	42.

5. To find the general solution of all these problems we consider its second form, with one change. We allow the letters to be in any order, so that now $((ab)(cd))$ is considered as distinct from $((ba)(cd))$ or from $((ab)(dc))$ or from $((ca)(bd))$, and so on.

Suppose now we can make A_n associations from n letters. How are A_n, A_{n-1} connected? If an association of $n-1$ letters is given, a new letter x can be inserted at the beginning, or at the end, or internally.

In the first two cases we get $(x(\ldots))$ and $((\ldots)x)$, the part $(\ldots)$ denoting any association of $n-1$ letters. These give then a contribution $2A_{n-1}$ to A_n.

To consider internal insertions, suppose that at some stage in the building up of the association of $n-1$ letters we have constructed $((P)(Q))$ where P, Q are letters or associations of some of the $n-1$ letters. Thus P might be $(((pq)(rs))t)$, Q might be $(u(vw))$.

Then x can be inserted in four ways

$$(((x(P))Q), \quad ((((P)x)Q), \quad ((P)(x(Q)), \quad ((P)((Q)x)).$$

But if we have $n-1$ letters the association is formed by making $n-2$ successive binary associations, and thus we get $4(n-2)$ associations, when x is inserted, for each one of the old, i.e. in all $4(n-2)A_{n-2}$ associations.

Hence, finally, the total number of associations of n letters is

$A_1 = 1$
$A_2 = 2$
$A_3 = 12$

$$A_n = 2A_{n-1} + 4(n-2)A_{n-1}$$

$$A_n = (4n-6)A_{n-1} = (4n-6)(4n-10)A_{n-2}$$

$$= (4n-6)(4n-10)(4n-14)\ldots(4r-2)A_r$$

But $A_1 = 1$. Hence

$$A_n = 2 \,.\, 6 \,.\, 10 \ldots (4n-6) = 2^{n-1}.\, 1 \,.\, 3 \,.\, 5 \ldots (2n-3)$$

Now in problem 2 the order of the letters was fixed. Hence to obtain the number B_n of solutions for that problem we divide A_n by $n!$

$$B_n = \frac{2^{n-1}(2n-2)!}{n!\, 2 \,.\, 4 \,.\, 6 \ldots (2n-2)} = \frac{(2n-2)!}{n!(n-1)!}$$

This then is the number of ways of dissecting an $\overline{n+1}$—gon as required in problem 1, and all the problems are solved e.g. $n = 6$ gives $B_n = 42$.

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