Neil Sloan 2C-363 Aug. 21

Neil,

Thanks for looking at this.

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Aug 12,

Dear Andrew,

I suggest that you might be interested in reading through the proof of the theorem — it is a striking example of the power of the technique of generating functions in solving recurrence relations. There are many manipulations left out, so it might take a bit of effort. I also highly recommend the first 18 pages of Riordan's book which I discovered just this last month. My question to your friends — Is the combinatorial result of interest, and if so, where might it be published?

The Euler characteristic for a finite cell complex of dimension n having v_r r-cells, $0 \le r \le n$, is the alternating sum $\sum_{r=0}^{n} (-1)^r v_r$. It is a topological invariant, it generalizes Euler's formula $V-E+F=2$ for polyhedra, and its vanishing for spheres of only odd dimension is the underlying reason that vector fields exist only for these spheres.

Yours, Bill

Partial Bracketings and Stasheff's K_n W. Butler, Aug. 1974

Bracketing a sequence of n letters leads, for $n = 2, 3, 4$, to the following geometric objects:

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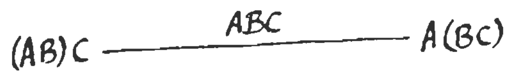

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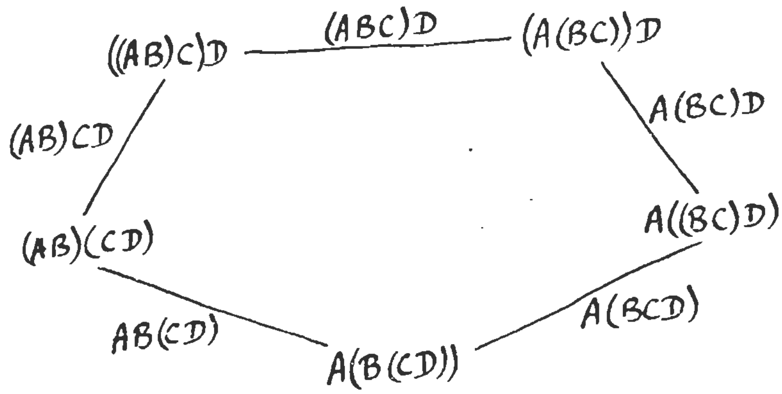

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Stasheff [1] has exhibited a sequence of cell complexes K_n, $n \geq 2$, where K_n is of dimension $n-2$ and the i-cells of K_n, $0 \leq i \leq n-2$, are indexed by all possible ways of "meaningfully" inserting $n-2-i$ sets of brackets in a sequence of n letters. (K_0, K_1 and K_2 are respectively a point, a segment and a pentagon as displayed above.) Moreover, Stasheff proves (Prop 3 of [1]) that K_n is homeomorphic to the $(n-2)$-cube.

~~The purpose of this paper is first to count the i-cells of K_n, and secondly to state a conjecture on the symmetry of certain of the K_n.~~

Theorem. If a_n^r, $n \geq 2$, $0 \leq r \leq n-2$, is the number of ways of "meaningfully" inserting r sets of brackets in a sequence of n letters (or, equally, the number of $(n-2-r)$-cells of K_n), then

$$a_n^r = \frac{1}{n-1}\binom{n-1}{r+1}\binom{n+r}{r} \quad . \qquad (1)$$

The following table gives a_n^r for $2 \le n \le 7$, $0 \le r \le n-2$.

$n \backslash r$	0	1	2	3	4	5
2	1	·	·	·	·	·
3	1	2	·	·	·	·
4	1	5	5	·	·	·
5	1	9	21	14	·	·
6	1	14	56	84	42	·
7	1	20	120	300	330	132

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Note 1. Substituting $r = 0, 1, n-2$ in (1) gives, respectively,

$a_n^0 = 1$, $n \ge 2$, as one would expect,

$a_n^1 = \binom{n}{2} - 1$, $n \ge 3$, as one would also expect (also mentioned in [1], p. 277),

$a_n^{n-2} = \frac{1}{n}\binom{2n-2}{n-1}$, $n \ge 2$, as is well known (p. 74 of [2]).

~~Note 2. Consideration of (1) also gives that if p is prime, then p divides a_{p-1}^r for $r \ge 1$. (Indeed, p divides a_{pk}^r for $n \ge k$.)~~

Note 3. The identity

$$\sum_{r=0}^{n-2} (-1)^r a_n^r = (-1)^n \tag{2}$$

follows from Stasheff's result and the invariance of the Euler characteristic since K_n has a_n^{n-2-i} i-cells and the $(n-2)$-cube has Euler characteristic 1.
Using (1), one can prove (2) directly as a special case ($p = 1$, $m = n-2$) of the Vandermonde convolution variant ((5d), p. 10 of [3])

$$\binom{n-p}{m} = \sum_{r=0}^{m} (-1)^{r+m} \binom{n+r}{r}\binom{p+m}{p+r},$$

or as a special case ($i = r+1$, $j = n-1$, $k = n$) of the identity of Klee

$$\sum_{i=0}^{j} (-1)^i \binom{j}{i}\binom{i+j}{k} = (-1)^j \binom{j}{k-j}$$

(p. 518 of [4], p. 13 of [3]), or by comparing constant terms in the expansion of either side of the identity

$$[x - x(1-x^{-1})^{n-1}](1-x)^{-n-1} = x(1-x)^{-n-1} - (-1)^{n-1}x^{-n+2}(1-x)^{-2} .$$

(2) also follows (by induction on n) from (5) in the proof of the theorem below.

<u>Proof of Theorem</u>. For a given partial bracketing of a sequence of n letters, $n \geq 4$, consider the <u>left-most</u> (left-hand) <u>bracket</u> together with its corresponding right-hand bracket. This pair of brackets will divide the sequence into three parts of lengths k, l, m ($l \geq 2$, $k+m \geq 1$) where $k+l+m = n$:

$$A_1 \cdots A_k(A_{k+1} \cdots A_{k+l})A_{k+l+1} \cdots A_{k+l+m} . \tag{3}$$

To find a_n^r, $n \geq 4$, $r \geq 2$, that is to count r-bracketings of a sequence of length n, consider separately the four cases:

(i) $m=0$,

(ii) $m=1$,

(iii) $m \geq 2$, $A_{k+l+1} \cdots A_{k+l+m}$ is not enclosed in brackets,

(iv) $m \geq 2$, $A_{k+l+1} \cdots A_{k+l+m}$ is enclosed in brackets.

Define b_n^r, $n \geq 2$, $0 \leq r \leq n-2$, to be the number of ways of first dividing a sequence of $n+2$ letters into two parts each of length at least 2, and then inserting a <u>total</u> of r sets of brackets in the two separate sequences. Next, stipulate that

$$a_n^r = 0 \quad \text{and} \quad b_n^r = 0 \quad \text{if} \quad r < 0 \quad \text{or} \quad r > n-2 \quad \text{or} \quad n < 2 . \tag{4}$$

(Thus $a_0^0 = 0$, $a_1^0 = 0$, contrary to how one might be inclined to interpret them.) One then has the recurrence relation

$$a_n^r = 2\sum_{i=2}^{n-1} a_i^{r-1} + \sum_{i=2}^{n-2} b_i^{r-1} + \sum_{i=2}^{n-2} b_i^{r-2} , \quad n \geq 0, \ r \geq 1 . \tag{5}$$

The first term comes from cases (i) and (ii) where i represents ℓ in (3), and the second and third terms come from cases (iii) and (iv) respectively where i represents $\ell+m-2$ in (3). For $r=1$, there is neither a case (iv) nor a third term in (5). For $n=3$, there are only cases (i) and (ii), and only the first term of (5) is not zero. For $n=2$, the right side of (5) is zero, and the left side is not zero only if $r=0$ for which no claim is made. For $n=0,1$, both sides of (5) are always zero. One also has the recurrence relation

$$b_n^r = \sum_{\substack{p+q=n+2\\ p,q\geq 2}} \sum_{\substack{k+\ell=r\\ k,\ell\geq 0}} a_p^k a_q^\ell \quad . \quad n\geq 0,\ r\geq 0 . \tag{6}$$

For $n=0,1$, both sides of (6) are zero. Next, define the functions

$$A = A(x,y) = \sum_{r\geq 0}\sum_{n\geq 0} a_n^r x^n y^r ,$$

$$B = B(x,y) = \sum_{r\geq 0}\sum_{n\geq 0} b_n^r x^n y^r .$$

Multiplying both sides of (5) by $x^n y^r$, and summing over $n\geq 0$, $r\geq 1$, one obtains (using (4) and that $a_n^0 = 1$, $n\geq 2$)

$$A - \frac{x^2}{1-x} = \frac{2xy}{1-x}A + \frac{x^2 y}{1-x}B + \frac{x^2 y^2}{1-x}B \quad . \tag{7}$$

Similarly from (6), summing over $n\geq 0$, $r\geq 0$, one has

$$x^2 B = A^2 \quad . \tag{8}$$

(7) and (8) give

$$(y+y^2)A^2 + (2xy - 1 + x)A + x^2 = 0 \quad .$$

Solving for A (and choosing the solution giving positive a_n^r),

$$A = -\frac{x}{1+y} + \frac{1-x}{2(y+y^2)} - \frac{1-x}{2(y+y^2)}\left(1 - \frac{4x}{(1-x)^2}\,y\right)^{\frac{1}{2}}$$

$$= -\frac{x}{1+y} + \frac{1}{1+y}\left(\frac{x}{1-x} + \frac{x^2}{(1-x)^3}\,y + \cdots + \frac{1}{r}\binom{2r-2}{r-1}\frac{x^r}{(1-x)^{2r-1}}\,y^{r-1} + \cdots\right).$$

Since a_n^r is the coefficient of $x^n y^r$ in A, and since

$$x^r(1-x)^{1-2r} = \sum_{k\geq 0}\binom{2r-2+k}{k}x^{r+k},$$

it follows that for $n \geq 2$,

$$(-1)^r a_n^r = \sum_{p=0}^{r}(-1)^p\frac{1}{p+1}\binom{2p}{p}\binom{n-1+p}{n-1-p}. \qquad (9)$$

(1) follows from (10) by simple induction on r.

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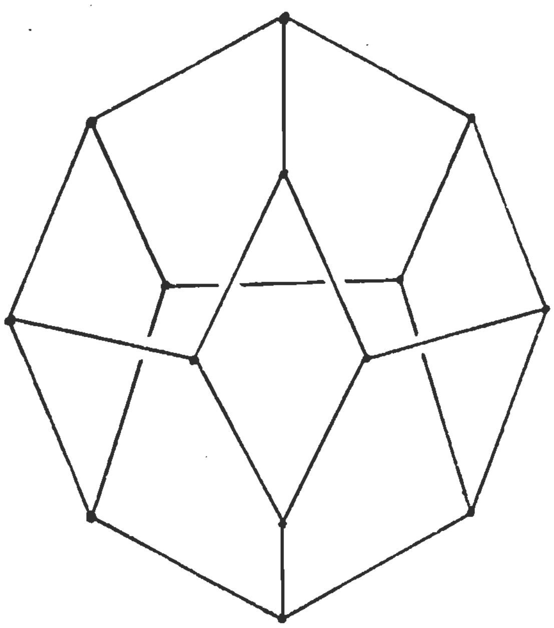
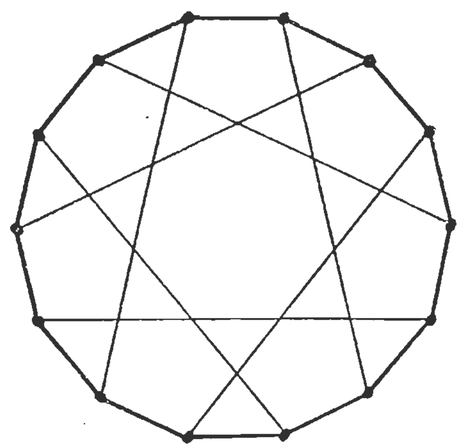
<u>Symmetries of K_n</u>. Determining the symmetries of K_n appears to be quite difficult. A sketch of K_5 follows. (See p.36, [5] for a flattened out picture with labelled vertices.)



Drawing K_6 is rather more complicated. K_6 has as 3-cells 7 K_5's and 7 $K_4 \times K_3$'s (pentagonal prisms), as 2-cells 28 K_4's (pentagons) and 28 $K_3 \times K_3$'s (squares), as 1-cells 84 K_3's (segments), and as 0-cells 42 K_2's (points). In general, one obtains the type of a cell inductively from its label. Choose any set of brackets — if the bracketing inside represents a cell of type A and the bracketing outside (considering the bracketed portion in question as a single letter) represents a cell of type B, then the complete bracketing represents a cell of type $A \times B$. The inclusion of cells corresponds to the omission of brackets in the corresponding labels. If K_6 is constructed explicitly, one notices that the 14 apexes of the 7 K_5's arrange themselves in a circuit represented here as the perimeter of the diagram



in which the 7 chords represent the pairing of apexes from the same K_5. The group of symmetries of this diagram is the dihedral group D_7, and each of these symmetries extends to a symmetry of the whole of K_6. This circuit acts as a spine for a distinguished Mobius strip of 7 pentagons joined to each other by non-adjacent edges and which are given at each stage a half turn. The 7 pentagonal prisms (which involve different pentagons) form a toroidal chain. They are joined by non-adjacent squares and are given at each stage a quarter turn in the same direction. A symmetry which sends each prism in the chain to the next sends each pentagon in the Mobius strip

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to the second succeeding one. The cells of K_6 of dimensions 0 through 3 divide themselves into orbits of either 7 or 14.

The following table gives the symmetry group of K_n, $2 \le n \le 6$.

K_2	K_3	K_4	K_5	K_6
1	2	D_5	D_3	D_7

The examples of K_4 and K_6, the observation of Note 2 (which is vacuous for $p < 5$) and the always present symmetry of order 2 $(n \ge 3)$ which turns labels around give credence to the following

End of delete

<u>Conjecture</u>. If $p \ge 5$ is prime, then K_{p-1} has D_p as symmetry group.

References

[1] J. D. Stasheff, Homotopy associativity of H-spaces, Trans. Am. Math. Soc. <u>108</u> (1963), 275 - 292

[2] C. L. Liu, Introduction to Combinatorial Mathematics, McGraw-Hill, New York, 1968

[3] J. Riordan, Combinatorial Identities, Wiley, New York, 1968

[4] V. Klee, A combinatorial analogue of Poincaré's Duality Theorem, Canadian J. Math <u>16</u> (1964), 517-531

~~[5] S. MacLane, Natural associativity and commutativity, Rice Univ. Studies 49 (1963), No. 4, 28-46~~