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AN INVESTIGATION OF SEQUENCES DERIVED FROM HOGGATT SUMS AND HOGGATT TRIANGLES

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1. INTRODUCTION

In a recent note [1], the authors discuss derivations of integer sequences called *Hoggatt Sums* and associated triangular arrays called *Hoggatt Triangles.* The nomenclature was proposed as a tribute to the late Verner Hoggatt, Jr. since the investigation and extension of an unpublicized conjecture of Hoggatt ultimately resulted in the above sums and triangles. In personal correspondence [2], Hoggatt conjectured that the third (counting as 0, 1, 2, 3, ...) right diagonal of Pascal's triangle could be used to determine the sequence of integers, S_0, S_1, S_2, ... S_m, ... , which are identically the Baxter permutation counts [3] of indices 0, 1, 2, ... , m, ... Hoggatt based his calculation algorithm for S_m on sums of products between third diagonal terms from Pascal's triangle and appropriately corresponding terms from a completed S_{m-1}. The authors' note [1] supplied the missing proof of Hoggatt's conjecture. Hoggatt's conjecture was then extended to include all right Pascal triangle diagonals indexed as 0, 1, 2, 3, ... , d, ... For each d, the set of S_m's became Hoggatt sums of order d, and the individual integers which sum to a particular S_m became row members of a triangular array called a Hoggatt triangle of order d. With the inclusion of d as a variable parameter, the numerical results of [1] can be interpreted as sequences of $(S_d)_m$'s with fixed index d and variable index m. For example, the Baxter permutation count values are Hoggatt sums of order three whose general sequence term is $(S_3)_m$. Sequences of Hoggatt sums follow a linear recursion which is *index-variant* in m, i.e., the calculation of $(S_d)_m$ for d fixed depends not only on previous members of the sequence but also depends on the value of m. Difference equations for this type of recursion are known to be difficult, if not impossible, to obtain by operational methods [4].

In the present investigation, Hoggatt sums and triangles are used to develop two new types of sequences. The first type (defined as *Hoggatt Second Kind Sequences*) consists of sequences of Hoggatt sums, $(S_d)_m$, with m fixed and d variable. The second type (defined as *Hoggatt Position Sequences*) consists of sequences of terms from the p^{th} positions of the m^{th} rows of d^{th} order Hoggatt triangles. Here p and m are fixed while d is the variable index. Both new sequences are *index-invariant* in d and belong to a class of recursions we have taken the liberty of calling "elementary". Of all the formally documented recursive sequences (see [5]), these stand out as the simplest. The index-invariant feature permits the use of operational methods to determine general forms for properties of the sequences once the degree, n, of recursion is established.

Since many of the results of this paper rely on numerical calculations, a word about calculation methods is in order. The calculations often involve extremely large but exact integers. There can be no rounding approximations! Except for lowest order calculations, hand methods or pocket calculators are out of the question. No matter how sophisticated they may be, conventional high-level computer languages eventually produce floating point and/or round-off

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errors. Fortunately, in recent years computer algebra systems such as MACSYMA, muMath, etc., [6] have been introduced. As well as performing complicated symbolic algebraic operations, the systems admit exact rational arithmetic with integers limited in size only by a well-managed available memory. We used a version of muMath on an AT clone personal computer. Thus, with size no obstacle, we were able to extend our integer use to meet the demands of almost any test situation.

2. BACKGROUND DETAILS

Prior to presenting our development, a very brief review of pertinent features of Hoggatt sums and triangles is essintial. The general form of Hoggatt's conjecture from [1], recast and extended for use herein, follows:

"Select the zeroth and d^{th} right diagonals of Pascal's triangle and let them be the zeroth and first right diagonals of a new triangle with as yet undetermined values for the remaining right diagonals. For m=2, 3, 4, ... in succession, compute the m^{th} row sum and m^{th} row individual entries as

$$(S_d)_m = 1 + \binom{m+d-1}{d}\left\{\frac{(R_{m-1})_0}{D_0} + \frac{(R_{m-1})_1}{D_1} + \ldots + \frac{(R_{m-1})_{m-1}}{D_{m-1}}\right\} \tag{1}$$

where the $(R_{m-1})_q$'s are the row integers from the previously completed $(m-1)$st row starting with q=0 on the left. The D_q's are the first diagonal integers starting with q=0 on the top right.

THEN, the m_{th} row sum, $(S_d)_m$, given by (1) is identically Hoggatt's sum with indices d and m. Morover,

$$1, \binom{m+d-1}{d}\left\{\frac{(R_{m-1})_0}{D_0}\right\}, \binom{m+d-1}{d}\left\{\frac{(R_{m-1})_1}{D_1}\right\}, \ldots, \binom{m+d-1}{d}\left\{\frac{(R_{m-1})_{m-1}}{D_{m-1}}\right\} \tag{2}$$

are the values of the row integers, P(m, p, d), of row m as assigned to positions 0, 1, ... , m."

An example taken from [1] illustrates for d=3, m=5 how the algorithm generates the fifth row terms from completed rows zero through four. The imcomplete Hoggatt triangle is shown as

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 4 & & 1 & & & \\ & & 1 & & 10 & & 10 & & 1 & & \\ & 1 & & 20 & & 50 & & 20 & & 1 & \\ 1 & & 35 & & x & & x & & x & & x \end{array} \tag{3}$$

The values chosen according to the conjecture result in the following generation of terms to complete the fifth row.

$$1,\ 35\left(\tfrac{1}{1}\right)= 35,\ \ 35\left(\tfrac{20}{4}\right)= 175,\ 35\left(\tfrac{50}{10}\right)= 175,\ 35\left(\tfrac{20}{20}\right)= 35,\ 35\left(\tfrac{1}{35}\right)= 1. \tag{4}$$

(Note that the zeroth and first terms of any row are always present.) The row sum $(S_3)_5$ = 1+35+175+175+35+1 = 422. A typical position term, say the fourth, is P(5,4,3) = 35.

A general form readily adaptable for computing Hoggatt sums is

$$1+\frac{\binom{m+d-1}{d}}{\binom{d}{d}}+\frac{\binom{m+d-1}{d}\binom{m+d-2}{d}}{\binom{d}{d}\binom{d+1}{d}}+\cdots+\frac{\binom{m+d-1}{d}\binom{m+d-2}{d}\cdots\binom{d+1}{d}\binom{d}{d}}{\binom{d}{d}\binom{d+1}{d}\cdots\binom{m+d-2}{d}\binom{m+d-1}{d}}, \tag{5}$$

which can be condensed to

$$(S_d)_m = 1 + \sum_{h=0}^{m-1}\prod_{k=0}^{h}\frac{\binom{m+d-1-k}{d}}{\binom{d+k}{d}}. \tag{6}$$

The value for P(m, p, d), the member in the p^{th} position of the m^{th} row of a d^{th} order Hoggatt triangle, is given by

$$P(m,\ p,\ d) = \prod_{h=0}^{p-1}\frac{\binom{m+d-1-h}{d}}{\binom{d+h}{d}}, \tag{7}$$

where P(m, 0, d) = 1 and p≤m. For example, P(5, 3, 4) is calculated as

$$\frac{\binom{8}{4}\binom{7}{4}\binom{6}{4}}{\binom{4}{4}\binom{5}{4}\binom{6}{4}} = 490. \tag{8}$$

Hoggatt triangles of orders 0 through 5, rows of index 0 through 6, are shown in Figure 1. The rectangular and circular enclosures are reserved for later use to illustrate properties of sets of P(m, p, d) entries for fixed m and p values.

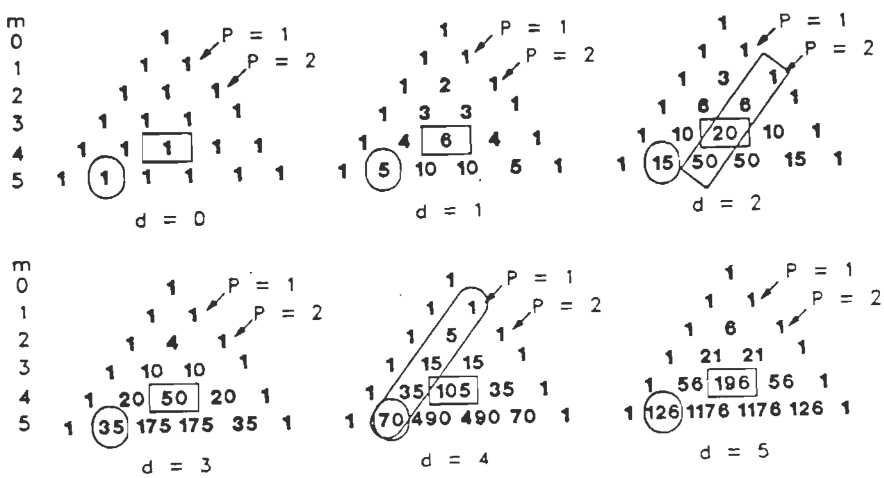

Figure 1. Partial Hoggatt Triangles, d = 0 to 5.

3. PROPERTIES OF ELEMENTARY RECURSIONS

In this section we define "elementary" recursions and develop general forms for their generating functions, homogeneous difference equations, and general sequence terms. If the order of recursion is n, the first n terms of the sequence can serve as initial conditions in a homogeneous equation or elsewhere as needed. A numerical algorithm by which the order, n, of any elementary recursion can be found is developed in the next section.

In determining the type and order of recursion in a sequence of integers, a convenient first trial is to calculate sucessive forward differences and hope to observe some unifying relation among the differences of various orders. (See Sloane [5], pp. 5-10.) The simplest of such observations occurs when the n^{th} differences are all zero. Regardless of order, these recursions are defined herein as "elementary". Such recursions are linear and index-invariant. Once the order of an elementary recursion is determined, its generating function, homogeneous difference equation, and general sequence term can be found from regular and predictable procedures.

As an illustration of the procedures, consider the sequence for $(S_d)_3$ d = 0, 1, 2, . . . (as calculated from (6)). For difference representation, Δ_d^0 replaces $(S_d)_3$. The partial triangle of differences becomes

d	Δ_d^0	Δ_d^1	Δ_d^2	Δ_d^3
0	4			
		4		
1	8		2	
		6		0
2	14		2	
		8		0
3	22		2	
		10		0
4	32		2	
		12		
5	44			

(9)

For local computations to follow, $(S_d)_3 = \Delta_d^0 = a_d$. We have $\Delta_d^3 = 0$, $\Delta_{d+1}^2 - \Delta_d^2 = 0$, $\Delta_{d+2}^1 - 2\Delta_{d+1}^1 + \Delta_d^1 = 0$, and $\Delta_{d+3}^0 - 3\Delta_{d+2}^0 + 3\Delta_{d+1}^0 - \Delta_d^0 = 0$. This leads to the homogeneous difference equation

$$a_{d+3} - 3a_{d+2} + 3a_{d+1} - a_d = 0. \tag{10}$$

The z-transform [4] of both sides of (10) yields

$$\left\{z^3\left[(a_d)-a_0\right]-z^2a_1-za_2\right\}-3\left\{z^2\left[Z(a_d)-a_0\right]-za_1\right\}+3\left\{z\left[Z(a_d)-a_0\right]\right\}-Z(a_d) = 0, \tag{11}$$

where a_0, a_1, a_2 are the initial conditions for (10). After factoring out $Z(a_d)$ and rearranging terms, we have the closed form of $Z(a_d)$ which is also the generating function in powers of $1/z$ of the sequence. $Z(a_d)$ becomes

$$Z(a_d) = \frac{z^3a_0+z^2(a_1\text{-}3a_0)+z(a_2\text{-}3a_1+3a_0)}{(z\text{-}1)^3} = \frac{4z^3\text{-}4z^2+2z}{(z\text{-}1)^3}. \tag{12}$$

(For those who wish a more conventional generating function form, $1/z$ can be replaced by x.) One way of getting a general sequence term is to find the inverse z-transform of (12) using an inversion integral [4] whose integrand is $Z(a_d)$ multiplies by z^{d-1}. The inversion integral is a contour integral in the z-plane where the contour can be taken as any counter-clockwise encirclement of the origin in the finite z-plane which completely encloses the unit circle without touching it.

$$a_d = \frac{1}{2\pi i}\oint \frac{\{z^3a_0+z^2(a_1\text{-}3a_0)+z(a_2\text{-}3a_1+3a_0)\}z^{d-1}\,dz}{(z\text{-}1)^3} \tag{13}$$

The general term, a_d, is equal to the residue of the integrand in the third order pole at $z = 1$. The integrand can be expanded in a Laurent expansion about the third order pole at $z = 1$. By dividing out he pole factor $(z\text{-}1)^3$, the Laurent expansion assumes the form shown below where the braced ({ }) portion is the numerator of the integrand with no singularities at $z = 1$. Hence the braced portion can be expanded in a Taylor's series about $z = 1$.

$$\frac{1}{(z\text{-}1)^3}\left\{\text{xxxx} + \text{xxxx}(z-1) + \text{Residue}(z-1)^2 + \text{xxxx}(z-1)^3 + \ldots\right\} \tag{14}$$

The only coefficient of interest to us in the Taylor expansion is the coefficient of $(z-1)^2$ because it is the coefficient of $1/(z-1)$ in the <u>Laurent</u> expansion of the integrand and, hence, the desired residue. A convenient way to avoid the differentiations of Taylor expansions is to shift plane origins by letting $(z-1) = W$, replace z by $(1+W)$, and after binomial expansion, pick the total coefficient of W^2 as the residue. The numerator of the integrand in W is

$$(1+W)^{d+2}a_0 + (1+W)^{d+1}(a_1-3a_0) + (1+W)^d(a_2-3a_1+3a_0). \tag{15}$$

The coefficient of W^2 in (15) becomes

$$a_d = \binom{d+2}{2}a_0 + \binom{d+1}{2}(a_1-3a_0) + \binom{d}{2}(a_2-3a_1+3a_0), \tag{16}$$

or, alternately,

$$a_d = \frac{(d+2)^{(2)}a_0 + (d+1)^{(2)}(a_1\text{-}3a_0) + (d)^{(2)}(a_2\text{-}3a_1+3a_0)}{2!}. \tag{17}$$

where $(d+2)^{(2)}$, for example, is the partial factorial $(d+2)(d+1)$, etc. For the numerical example, substitution leads to

$$a_d = (S_d)_3 = d^2 + 3d + 2. \tag{18}$$

Although the numerical results of (10), (12), and (18) serve as satisfying computational examples, (10) through (17) suggest what similar expressions for other specific orders and even the general order, n, might look like. For the <u>third</u> order case presented above, a guiding "threeness" is very evident in the expressions. As difference triangles are extended to higher orders, it becomes evident that for elementary recurisons of order n, it is Δ_d^n which first becomes zero and that n sequence tersm, a_0 through a_{n-1} are necessary to complete the triangle. The general homogeneous difference equation becomes

$$\sum_{r=1}^{n}(-1)^r\binom{n}{r}a^{d+n-r} = a^{d+n} - \binom{n}{1}a^{d+n-1} + \binom{n}{2}a^{d+n-2} + \cdots + (-1)^n\binom{n}{n}a^d = 0. \tag{19}$$

Application of the z-transform to (19) yields the general generating function

$$Z(a_d) = \frac{z^n a_0 + z^{n-1}\left(a_1 - \binom{n}{1}a_0\right) + z^{n-2}\left(a_2 - \binom{n}{1}a_1 + \binom{n}{2}a_0\right) + \cdots + z\left\{a_{n-1} - \binom{n}{1}a_{n-2} + \cdots + (-1)^{n-1}\binom{n}{n-1}a_0\right\}}{(z-1)^n} \tag{20}$$

Applying the inversion integral to (20), there results two equivalent forms of the general sequence term for elementary recursions of order n,

$$a_d = \left[\binom{d+n-1}{n-1}a_0 + \binom{d+n-2}{n-1}\left\{a_1 - \binom{n}{1}a_0\right\} + \binom{d+n-3}{n-1}\left\{a_2 - \binom{n}{1}a_1 + \binom{n}{2}a_0\right\} + \cdots + \binom{d}{n-1}\left\{a_{n-1} - \binom{n}{1}a_{n-2} + \cdots + (-1)^{n-1}\binom{n}{n-1}a_0\right\}\right], \tag{21}$$

$$a_d = \frac{\left[(d+n-1)^{(n-1)}a_0 + (d+n-2)^{(n-1)}\left\{a_1 - \binom{n}{1}a_0\right\} + \binom{d+n-3}{n-1}\left\{a_2 - \binom{n}{1}a_1 + \binom{n}{2}a_0\right\} + \ldots + (d)^{(n-1)}\left\{a_{n-1} - \binom{n}{1}a_{n-2} + \ldots + (-1)^{n-1}\binom{n}{n-1}a_0\right\}\right]}{(n-1)!} \tag{22}$$

Note that the example solutions are merely special cases of the above for $n = 3$. If n and a_0 through a_{n-1} of an elementary recursion are known, it is now possible to write the homogeneous difference equation, the generating function, and the general sequence term almost by inspection.

4. ALGORITHM FOR FINDING ORDER, n

The expressions for homogeneous difference equations, generating functions, and general sequence terms derived above, are worthless unless n and the first n sequence terms, a_0 through a_{n-1} are known. However, the primary purpose of this paper is to investigate the above properties in two types of elementary recursive sequences from Hoggatt sums and triangles in which n is not known. On the other hand, because of (6) and (7), all sequence terms, a_0 through a_{n-1} and beyond, are known for any n whatever. All that is needed is the m specification in (6) and the m, p specification in (7). If we hope to generalize n as a function of m (or m and p) through computer simulation, we must obtain n for a sufficiently large sampling involving m (or m and p).

Even though a recursion is known to be elementary, the difference triangle approach can become overwhelmingly tedious for large n and is not particularly suitable for machine computation. As a practical algorithm for finding n, consider the generating function (20) for any order n. When expanded, (20) is the open form of the z-transform, and infinite sequence in powers of $1/z$. This sequence can be found by dividing $(z\text{-}1)^n$ into the numerator of (20). Since the first n sequence coefficients, a_0 through a_{n-1} are presumed known for any chosen n, the quotient can be expressed as

$$a_0 + a_1/z + a_2/z^2 + \cdots + a_{n-1}/z^{n-1} + (-1)^{n-1} + \Big\{a_0 - \binom{n}{1}a_1 + \cdots + (-1)^{n-1}\binom{n}{n-1}a_{n-1}\Big\}/z^n + \cdots \tag{23}$$

The coefficient, a_n, of $1/z^n$ (in braces { }) is the first coefficient to depend on all n coefficients, a_0 through a_{n-1}. Because of recursion, all coefficients beyond and including a_n thereby depend on a_0 through a_{n-1}. For a_n in particular, we have

$$a_n = (-1)^{n-1}\Big\{a_0 - \binom{n}{1}a_1 + \binom{n}{2}a_2 + \ldots + (-1)^{n-1}\binom{n}{n-1}a_{n-1}\Big\}. \tag{24}$$

If a_n is subtracted from both sides of (24), the result, <u>regardless of $(-1)^{n-1}$</u>, is

$$\Big\{a_0 - \binom{n}{1}a_1 + \binom{n}{2}a_2 + \cdots + (-1)^n a_n\Big\} = 0. \tag{25}$$

For our two types of sequences, the value of n is not known but as many of a_0, a_1, a_2, $\cdots$ as are needed are available. Start with n=1 and test (25) for { } = 0. If { } $\neq$ 0, then try successively larger trial n's. The smallest n which satisfies (25) is the order, n, of the elementary recursion. It can be shown that any larger trial n also satisfies (25), so that a wide variety of bracketing steps could be employed to find n. In a simple illustration of the algorithm as used herein, sequence terms with the sign of every odd-indexed term reversed are multiplied by binomial coefficeints corresponding to a test n. The products are added and the sum tested for zero. If the sum is not zero, n is incremented and the process repeated as required using the next order of binomial coefficeints. An example of a test which results in the recursion order seven for the sequence for $(S_d)_5$ is shown in (26).

n=5

$+a_0$	6x 1=	6
$-a_1$	−32x 5=	−160
$+a_2$	132x10=	1320
$-a_3$	−422x10=	−4220
$+a_4$	1122x 5=	5610
$-a_5$	−2606x 1=	−2606
	TOTAL=	−50

n=6

$+a_0$	6x 1=	6
$-a_1$	−32x 6=	−192
$+a_2$	132x15=	1980
$-a_3$	−422x20=	−8440
$+a_4$	1122x15=	16830
$-a_5$	−2606x 6=	−15636
$+a_6$	5462x 1=	5462
	TOTAL=	10

n=7

$+a_0$	6x 1=	6
$-a_1$	−32x 7=	−224
$+a_2$	132x21=	2772
$-a_3$	−422x35=	−14770
$+a_4$	1122x35=	39270
$-a_5$	−2606x21=	−54726
$+a_6$	5462x 7=	38234
$-a_7$	−10562x 1=	−10562
	TOTAL=	0

(26)

5. HOGGATT SEQUENCES OF THE SECOND KIND

Since the original Hoggatt sums were sequences of $(S_d)_m$ with m as the variable index and d fixed, it seemed logical to name sequences of $(S_d)_m$ in which d is the variable index and m is fixed, Hoggatt sequences of the second kind. The second kind sequences are much easier to work with since the recursion is elementary. Examples of several second kind sequences calculated using (5) or (6), are listed in (27). Shown also are the values of recursion order, n, versus m. The values of n were found using members of the second kind sequences and the order algorithm discussed in the previous section.

d	$(S_d)_0$	$(S_d)_1$	$(S_d)_2$	$(S_d)_3$	$(S_d)_4$	$(S_d)_5$	$(S_d)_6$	$(S_d)_7$	
0	1	2	3	4	5	6	7	8	
1	1	2	4	8	16	32	64	128	
2	1	2	5	14	42	132	429	1430	A108
3	1	2	6	22	92	422	2074	10754	(27) A1181
4	1	2	7	32	177	1122	7898	60398	A5362
5	1	2	8	44	310	4606	25202	272582	A5363
6	1	2	9	58	506	5462	70226	1038578	A5364
7	1	2	10	74	782	10562	175826	3457742	A5365
8	1	2	11	92	1157	19142	403691	10312304	A5366
9	1	2	12	112	1652	32892	862864	28066040	
10	1	2	13	134	2290	54056	1736737	70702634	
m	0	1	2	3	4	5	6	7	
n	1	1	2	3	5	7	10	13	

It is suggested by (27) that n is a function of m and, except for m = 0, the order is never less than m. This makes m a good first candidate for a trial n in the order algorithm. However, before any explicit relationships can be found, more comparisons must be observed. Through extensive use of muMath, the order algorithm, and values generated by (5) or (6), the following tabulation was found:

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m	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
n	1	1	2	3	5	7	10	13	17	21	26	31	37	43	50	57	65	73	82	91

(28)

The repeating differences observed between n values in the "order" sequence (28) suggest that two sequences exist, one for m odd and one for m even.

For m odd, let h = (m+1)/2 so that h varies as 1, 2, 3, ... as m takes on 1, 3, 5, ... The order calculation is so simple that the difference triangle method quickly shows $\Delta_h^3 = 0$ and thereby indicates an elementary third order recursion for the sequences of n for m odd. By observing the difference behavior, the missing "a_0" can be found and included. Now, by using $a_0 = 1$, $a_1 = 1$, $a_2 = 3$, and in (22) replacing "n" by three and d by h, the general sequence term for the odd m sequence appears as

$$\frac{(h+2)^{(2)} - 2(h+1)^{(2)} + 3(h)^{(2)}}{2!} = h^2 - h + 1. \tag{29}$$

When h is replaced by (m+1)/2, the general order value, n, for second kind sequences for odd m becomes

$$n = (m^2+3)/4. \tag{30}$$

A repeat of the process using m even and h $-$ m/2 yields

$$n = (m^2+4)/4. \tag{31}$$

By combining (30) and (31), the recursion order, n, for any m for second kind sequences of $(S_d)_m$ becomes

$$n = \frac{m^2 + 3.5 + (-1)^m(0.5)}{}. \tag{32}$$

Now, for any specified m, the exact number, n, of sequence terms, $(S_0)_m$, $(S_1)_m$, ... , $(S_{n-1})_m$, can be calculated from (5) or (6) to serve as initial constants for an almost at sight development of homogeneous difference equations [see (19)], generating functions in 1/z [see (20)], and general sequence term values [see (21) or (22)].

6. HOGGATT POSITION SEQUENCES

Expression (7) determines the value of the integer in the pth position of the mth row of a dth order Hoggatt triangle and assigns the ordered functional designation, P(m, p, d). The sequences formed by P(m, p, d) values with m and p fixed and variable d = 0, 1, 2, 3, ... are called Hoggatt position sequences. Fortunately, they exhibit elementary recursion so that the methods of previous sections can be used to again obtain general expressions for recursion order, n, homogeneous difference equations, generating functions, and sequence terms. As before, the ultimate goal is the ability to write the expressions almost by inspection once n and the first n sequence terms are available.

As illustrations, portions of position sequences can be observed in Figure 1. If m = 2, p = 2, the integers of sequence 1, 6, 20, 50, 105, 196, ... are set off by rectangular enclosures, while for m = 5, p = 1 oval enclosures are used to enclose the sequence 1, 5, 15, 20, 70, 126, ... While the definition of position sequences is global with respect to d, an interesting feature is that any particular position sequence has an equal, matching diagonal sequence completely contained in some Hoggatt triangle. This property is identified in Figure 1 for the above position sequences for the diagonal sequence in triangle two by a rectangular enclosure and for triangle four by an oval enclosure. These phenomena have the practical aspect of making two observational and calculation approaches available.

A position sequence term can be distinguished from a diagonal sequence term by choosing $P(m_n, P_h, d)$ for the position term and $P(m, p_d, d_d)$ for the diagonal term where m_h, p_h, p_d and d_d are constants. Because of symmetry of Hoggatt triangles about a central, "vertical" axis [1], we restrict our position sequence computations to $0 \leq p \leq m/2$ for m even and to $0 \leq p \leq (m-1)/2$ for m odd. The matching diagonal sequences proceed from upper right to lower left.

At the integer of intersection of a position sequence with its matching equal diagonal sequence, $m = m_h$, $d = d_d$, and $p_h = p_d$. Since p_h and p_d are constants, they can be replaced by a general constant, p. Also the integer of intersection must be the d_d term of each sequence since the intersection takes place in the d_dth triangle. Counting back along the diagonal sequence from the intersection integer determines the starting m of the diagonal sequence as $m_h - d_d$. Since the diagonal sequence must start on the extreme right of the d_dth triangle, the m value for that row of the d_dth triangle fixes p as

$$p = m_h - d_d, \tag{33}$$

with p also the starting value of m of the diagonal sequence. The order of the triangle in which intersection takes place is, thereby,

$$d_d = m_h - p, \tag{34}$$

where m_h and p are established position sequence constants. Correspondingly, if a diagonal sequence is already specified, the m_h of its matching position sequence is given as

$$m_h = d_d + p. \tag{35}$$

With release of the position restriction and with p taken as $m/2 \leq p \leq m$ for m even, or $(m-1)/2 \leq p \leq m$ for m odd, it is seen that the matching diagonal sequences start on the upper left and proceed to the lower right.

As an illustration of (35), the diagonal shown on Figure 1 for d = 4 starting with 1, 35, 490, ... has p = 3, d_d = 4 which results in m_h = 7. The matching position sequence representation is P(7, 3, d) which is beyond the limits of Figure 1. However, calculations made using (7) verify the first few terms of the position sequence as 1, 35, 490, 4116, 24696, 116424, 457380, ...

The remaining study of position sequences is devoted to finding the general expression for recursion order, n, as a function of m and p. Through use of the order algorithm, the orders of position sequences P(m, p, d) were calculated in m increments from zero through nine with p subincrements (for each m) from zero through m/2 for m even and through $(m-1)/2$ for m odd. The full range of p was not needed because of Hoggatt triangle symmetry. The values of recursion order, n, for the ranges of m and p are given in (36).

p \ m	0	1	2	3	4	5	6	7	8	9
0	1	1	1	1	1	1	1	1	1	1
1		1	2	3	4	5	6	7	8	9
2			1	3	5	7	9	11	13	15
3				1	4	7	10	13	16	19
4					1	5	9	13	17	21
5						1	6	11	16	21
6							1	7	13	19
7								1	8	15

← n values (36)

Each of the sequences of n values with m assuming the rôle of index is seen to have an elementary recursion of order two, with the first difference identically p regardless of m. The "a_0, a_1" first two terms are always 1 and 1 + p. However, the starting index for each sequence changes with m which is not consistent with our methods. To force a starting index of zero for each sequence, the index must be adjusted to m − p. When this is done, replacement of d by m − p, replacement of "n" by 2, substitution of 1 and 1 + p for a_0 and a_1 in (22) yields the general n as

$$n = (m-p+1)a_0 + (m-p)(a_1-2a_0) = (m-p+1) + (m-p)(p-1) = (m-p)p + 1. \qquad (37)$$

Thus the order of any Hoggatt position sequence P(m, p, d) can be specified directly in terms of its fixed m and p parameters. To further aid in visual identification, (37) is arranged in (38) as an "order" triangle, allowing the order, n, of a position sequence, P(m, p, d), to be selected at sight from the mth row, pth position integer.

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m										
0	1									
1	1	1								
2	1	2	1							
3	1	3	3	1						
4	1	4	5	4	1					
5	1	5	7	7	5	1				
6	1	6	9	10	9	6	1			
7	1	7	11	13	13	11	7	1		
8	1	8	13	16	17	16	13	8	1	
9	1	9	15	19	21	21	19	15	9	1 .

(38)

Now that the recursion order for Hoggatt position sequences is established as (m−p)p + 1, it is possible to state a set of initial conditions, the homogeneous difference equation, the generating function in 1/z, and the general sequence term for any Hoggatt position sequence.

The Hoggatt position sequences seem to have some value in counting specialized distributions of objects. As a parting, proverbial "exercise for the reader," deduce the P(m, p, d) position sequence for which the a_r [or P(m, d, r)] term answers the following:

In how many ways can r indistinct objects be distributed in six distinct cells with five of the cells permitted any occupancy (including none) but with the sixth cell permitted either no occupancy or single occupancy?

7. CONCLUSIONS

We have introduced two new classes of sequences, Hoggatt second kind sequences and Hoggatt position sequences. Even though we have classified the sequences as having the most elementary form of recursion, each sequence has exhibited an interesting behavior. An order algorithm developed for use with the sequences provided data for completing the specifications of homogeneous difference equations, generating functions, and formulas for general terms for the sequences.

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