Math Gazette 21 (1937)

NON-ASSOCIATE POWERS AND A FUNCTIONAL EQUATION.

By I. M. H. Etherington.

Several writers* have studied algebras in which multiplication is non-associative, that is, $x \,.\, yz \neq xy \,.\, z$. It is necessary† in a non-associative algebra to distinguish the possible interpretations of a power x^n. In a non-commutative non-associative algebra x^2 is unique: x^3 can mean xx^2 or x^2x; x^4 can mean $x \,.\, xx^2$, $x \,.\, x^2x$, x^2x^2, $xx^2 \,.\, x$ or $x^2x \,.\, x$; x^5 has 14 interpretations; x^6 has 42; and so on. In a commutative non-associative algebra, the possible interpretations are fewer: x^3 is unique; x^4 can mean xx^3 or x^2x^2; x^5 can mean $x \,.\, xx^3$, $x \,.\, x^2x^2$ or x^2x^3; x^6 has 6 interpretations; and so on. The problem considered here is: how many meanings are there for x^n (A) in a general non-commutative non-associative algebra? (B) in a general commutative non-associative algebra? The answer to (A) is $\dfrac{2(2n-3)!}{n!(n-2)!}$. I am not able to find any such simple formula for (B).

It may be observed that (A) is equivalent to asking how many meanings there are in general in ordinary algebra for $x_1 : x_2 : \dots : x_n$, where each colon stands for any non-commutative non-associative process, such as subtraction, division or exponentiation.

(A) *The number of interpretations of x^n in a general non-commutative non-associative algebra.*

Let a_n denote the required number of interpretations. Since x^n can arise in any of the ways xx^{n-1}, x^2x^{n-2}, x^3x^{n-3}, ..., $x^{n-1}x$, we have at once for $n>1$

$$a_n = a_1a_{n-1} + a_2a_{n-2} + a_3a_{n-3} + \dots + a_{n-1}a_1.$$

The left-hand side is the coefficient of x^n in

$$F(x) = a_1x + a_2x^2 + a_3x^3 + \dots \textit{ ad inf.}:$$

while the right-hand side, for $n>1$, is the coefficient of x^n in $\{F(x)\}^2$. Therefore for sufficiently small values of x (assuming the absolute convergence of the series for $F(x)$)

$$F(x) = x + \{F(x)\}^2. \qquad \dots\dots\dots\dots(1)$$

whence $$F(x) = \tfrac12 \pm \tfrac12\sqrt{1-4x}.$$

The minus sign must be taken because $F(0)=0$. So

$$F(x) = \tfrac12 - \tfrac12\sqrt{1-4x} = \tfrac12 - \tfrac12\left(1 - 2x + \frac{\frac12(-\frac12)}{2!}\,.\,16x^2 - \dots\right). \quad \dots\dots(2)$$

the assumption of absolute convergence being justified if $|x| < \frac14$.

* *E.g.* De Morgan, *Trans. Camb. Phil. Soc.*, 8 (1844), 241; Dickson, *Trans. Amer. Math. Soc.*, 13 (1912), 60. Jordan, *Göttingen Nachr.*, 1932, 569; 1933, 209, deals with an application to Quantum Mechanics. A forthcoming paper by the author on Genetics uses commutative non-associative algebras.

† Though not in the algebras considered by Jordan.

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Hence a_n = coefficient of x^n

$$= -\frac{1}{2\,.\,n!}\,.\,\frac12\left(-\frac12\right)\left(-\frac32\right)\dots\left(-\frac{2n-3}{2}\right).\,(-4)^n.$$

If $n>1$, this $=[1\,.\,3\,.\,5\,.\,\dots\,(2n-3)]2^{n-1} \div n! = \dfrac{2(2n-3)!}{n!(n-2)!}$,

a formula which is easily checked for small values of n.

(B) *An attempt to find the number of interpretations of x^n in a general commutative non-associative algebra.*

The attempt fails because the general term of the expansion corresponding to (2) cannot be written down. However, as the functional equation (4) which replaces (1) is interesting on its own account, it is perhaps worth while giving the details.

Let b_n denote the required number of interpretations, and let

$$f(x) = -1 + b_1x + b_2x^2 + b_3x^3 + \dots \textit{ ad inf.} \qquad \dots\dots\dots\dots(3)$$

The insertion of -1 simplifies the functional equation which arises.

Since $b_n < a_n$, the series is absolutely convergent when $|x| < \frac14$; so $f(x)$ may be formally squared.

Now x^n can be formed in any of the ways xx^{n-1}, x^2x^{n-2}, x^3x^{n-3}, ... This series terminates with $x^{\frac{n-1}{2}}x^{\frac{n+1}{2}}$ if n is odd and >1, with $x^{\frac{n}{2}}x^{\frac{n}{2}}$ if n is even. Hence, for $n>1$,

$$b_{2n-1} = b_1b_{2n-2} + b_2b_{2n-3} + \dots + b_{n-1}b_n$$
$$= \text{half the coefficient of } x^{2n-1} \text{ in } \{f(x)+1\}^2;$$
$$b_{2n} = b_1b_{2n-1} + b_2b_{2n-2} + \dots + b_{n-1}b_{n+1} + \tfrac12 b_n(b_n+1)$$
$$= \text{half the coefficient of } x^{2n} \text{ in } \{f(x)+1\}^2 + f(x^2).$$

So the expansions of the functions $f(x)$ and $\frac12[\{f(x)+1\}^2 + f(x^2)]$ agree from the terms in x^3 onwards. These expansions begin respectively $-1 + x + x^2 + x^3 + \dots$ and $-\frac12 + x^2 + x^3 + \dots$. It follows that

$$f(x) + 1 - x = \tfrac12[\{f(x)+1\}^2 + f(x^2)] + \tfrac12,$$

that is, $$\{f(x)\}^2 + f(x^2) + 2x = 0, \qquad \dots\dots\dots\dots(4)$$

or $$f(x) = -\sqrt{-2x - f(x^2)}. \qquad \dots\dots\dots\dots(5)$$

When x is small, we know from (3) that, to a first approximation, $f(x) = -1 = f(x^2)$; substituting this in the right-hand side of (5), we get the second approximation $f(x) = -\sqrt{-2x+1}$, $f(x^2) = -\sqrt{-2x^2+1}$; similarly the third approximation is $f(x) = -\sqrt{-2x+\sqrt{-2x^2+1}}$, and so on. Hence

$$f(x) = \lim_{N\to\infty} -\sqrt{-2x+\sqrt{-2x^2+\sqrt{-2x^4+\dots+\sqrt{-2x^{2^N}+1}}}}, \quad \dots\dots(6)$$

where each $\surd$ sign covers all that follows it.

Abandoning the attempt to find b_n explicitly by expanding this generating function, let us consider the function itself and the various other functions which satisfy equation (4) but not (3).

Suppose we assign an arbitrary value to $f(x_0)$. Then from (4) we can calculate in succession $f(x_0^2), f(x_0^4), f(x_0^8), \ldots$; and the (possibly infinite) limit of this sequence gives either $f(0)$ or $f(\infty)$ according as $|x_0| <$ or > 1. This shows that the functions which satisfy (4) are of two kinds, those for which $f(0)$ exists and those for which $f(\infty)$ exists. In fact it will be found that no curve of the system $y=f(x)$ crosses the boundaries $x=\pm 1$.

Consider first the functions for which $f(0)$ exists. These are of three types: for (4) gives $\{f(0)\}^2+f(0)=0$, so that

$$f(0)=0, \quad -1 \text{ or } -\infty. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

When x is small, let the first approximation be $f(x)=kx^a$.

Substituting in (5), we get as the second approximation

$$f(x)=-\sqrt{-2x-kx^{2a}}, \quad f(x^2)=-\sqrt{-2x^2-kx^{4a}}.$$

For $f(x^2)$ to be real the term in x^{4a} must predominate, so $\alpha<\frac{1}{2}$; and in order that the two approximations for $f(x)$ should agree, $k=-1$. Iterating as before,

$$f(x)=\lim_{N\to\infty} -\sqrt{-2x+\sqrt{-2x^2+\sqrt{-2x^4+\ldots+\sqrt{-2x^{2^N}+x^{2^{N+1}a}}}}}, \ldots(8)$$

where each $\sqrt{\ }$ sign covers all that follows it. The three types (7) correspond to $\frac{1}{2}>\alpha>0$, $\alpha=0$, $\alpha<0$; and $\alpha=0$ gives (6).

The functions for which $f(\infty)$ exists are found in precisely the same way. Assuming $f(x)\sim kx^a$ for large x, we reach by iteration the same formula (8), but now $\alpha>\frac{1}{2}$.

If $x>0$, the right-hand side of (8) has a factor $\sqrt{x}$. Writing $f(x)=Y\sqrt{x}$, the formula becomes

$$Y=-\lim_{N\to\infty}\theta^{N+1}(x^{2^{N+1}a-2^N}). \quad \ldots\ldots\ldots\ldots\ldots\ldots(9a)$$

where θ is the operation defined by $\theta t=\sqrt{-2+t}$. Inverting (9a),

$$x=\lim_{N\to\infty}(\phi^{N+1}Y)^{\frac{1}{2a-1}\cdot\frac{1}{2^N}}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(10a)$$

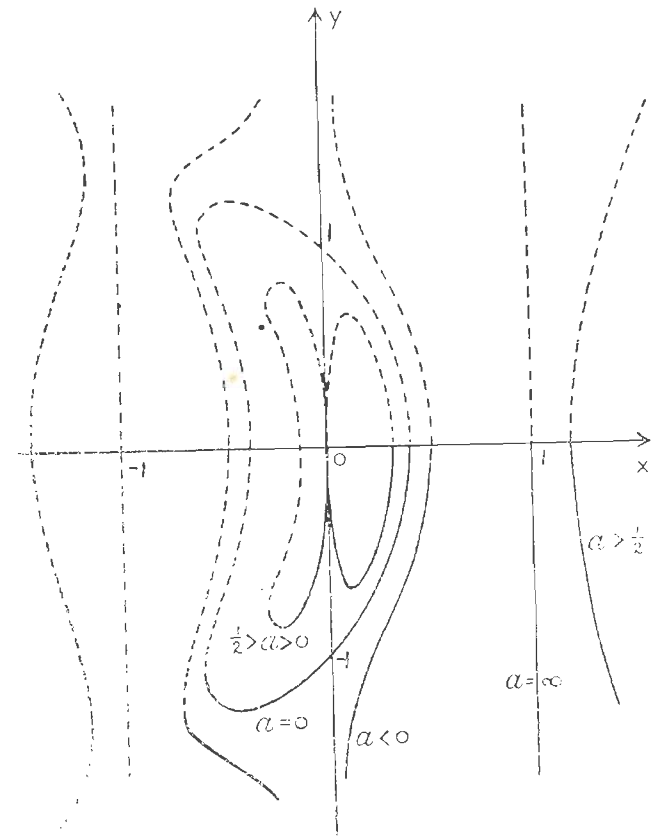
where $\phi t=t^2+2$. Similarly for $x<0$, writing $f(x)=Z\sqrt{-x}$,

$$Z=-\lim_{N\to\infty}\sqrt{2+\theta^N x^{2^{N+1}a-2^N}}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(9b)$$

$$x=\lim_{N\to\infty}\{\phi^N(Z^2-2)\}^{\frac{1}{2a-1}\cdot\frac{1}{2^N}}. \quad \ldots\ldots\ldots\ldots\ldots(10b)$$

The curves $y=f(x)$ can be traced from these formulae (8)-(10) combined with (4). The various types are shown by the continuous lines in the figure. When $\alpha<0$ or $>\frac{1}{2}$, the curve has two branches. If $f(x^2)$ in (4) is taken to mean "one value of $f(x^2)$", then all the radicals in (8) and (9) can be taken as "plus or minus". (Most of the radicals have to be taken positively to yield a real value of $f(x)$.)

With this generalisation the curves continue as shown by the dotted lines in the figure. The zeros of $f(x)$ can be calculated from equations (10) by putting $Y=0$, $Z=0$. These equations also show that



for $\alpha=\infty$, $x=\pm 1$; and that for $\alpha=\frac{1}{2}$ the curve becomes the line at infinity plus the origin. Actually, when $\alpha=\frac{1}{2}$, the process of iteration terminates and gives the solution $f(x)=kx^{\frac{1}{2}}$, where k is one of the (imaginary) roots of $k^2+k+2=0$. I. M. H. E.

1103. . . . We have had many earlier promises that the Government needed more time for the consideration of Mr. S——'s proposals—time that has been infinite. (In fact, it has had the latest proposals before it for nearly four months.)—*Manchester Guardian Weekly*, November 20, 1936, p. 402, c. 3 [Per Mr. J. W. Ashby Smith.]

Etherington, I. M. H., Postscript
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says:

"... results of my article on "Non Assoc Powers..." were obtained by Wedderburn with greater generality & rigour. See
— Ann Math 1922, vol 24.