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THE UNIVERSITY OF CALGARY

Department of Mathematics

DISSECTING A POLYGON INTO TRIANGLES

Richard K. Guy

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1. Introduction: This paper is an almost verbatim reprint of [44], but the opportunity has been taken to add Bill Brown's excellent bibliography [54] and to check the table of results against Motzkin's [36]. There was agreement, except for the error noted by Moon and Moser [50]. The note arose from a problem [43] taken from Polya [39], which may be expressed as: find D_n, the number of dissections of a convex polygon of n sides into $n-2$ triangles by drawing various sets of $n-3$ non-intersecting diagonals.

2. We give two arguments: First argument: Take any such dissection.

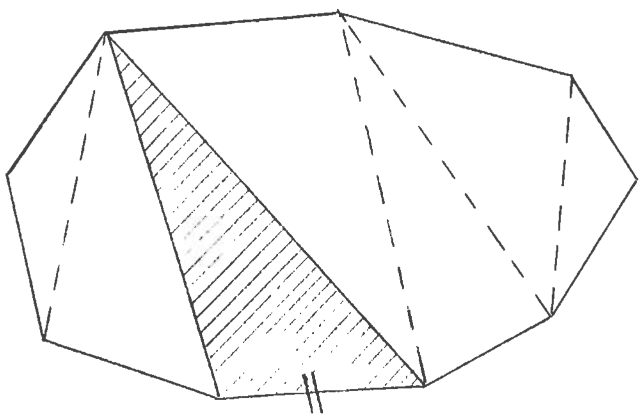

FIGURE 1

Fix a side (marked in figure 1). This determines a unique triangle of the dissection of whose sides this is one (shaded in figure). There remain two polygons of $r+1$ and $n-r$ sides, where r may be any number from 1 to $n-2$. The case $r=1$ giving a 'two-sided polygon' corresponds to that in which the shaded triangle has *two* sides in common with the original polygon. In what follows D_2 is a convenient way of writing unity.

The two polygons may be dissected in $D_{r+1}D_{n-r}$ ways, so that all dissections of the original polygon are given by

$$\sum_{r=1}^{n-2} D_{r+1}D_{n-r},$$

i.e. $D_n = D_2D_{n-1} + D_3D_{n-2} + D_4D_{n-3} + \ldots + D_{n-2}D_3 + D_{n-1}D_2$. (1)

Second argument: We may draw a diagonal in the original polygon in $\frac{1}{2}n(n-3)$ ways. For convenience, we will attach a direction to the diagonal, giving $n(n-3)$ possibilities. Take a dissection. Select a diagonal and attach a direction to it. This may be done in $(n-3)\times 2$ ways. This diagonal divides the polygon into two of $r+1$ and $n-r+1$ sides, where r is some number from 2 to $n-2$ inclusive. These may be dissected in $D_{r+1}D_{n-r+1}$ ways, and there are n diagonals which divide the polygon in each of these ways, so

$$2(n-3)D_n = n(D_3D_{n-1} + D_4D_{n-2} + \ldots + D_{n-1}D_3). \quad (2)$$

The first problem I have been unable to solve is to deduce (1) from (2) or vice versa without recourse to further geometrical argument, i.e. by algebra, or possibly analysis.

We next require a specific formula for D_n. If we *know* the answer

$$D_n = \frac{(2n-4)!}{(n-1)!(n-2)!} \quad (3)$$

then it is easy to prove by induction, since it is true for $n=2,3$, because $D_2=D_3=1$, and assuming it true for $n=2,3,\ldots r$, (1) gives

$$D_{r+1} = \frac{(2r-4)!}{(r-1)!(r-2)!} + \frac{2!}{1!2!}\,\frac{(2r-6)!}{(r-2)!(r-3)!} + \dots + \frac{(2r-4)!}{(r-1)!(r-2)!}$$

$$= 2^{r-2}\left\{\frac{1.3.5...2r-5}{(r-1)!} + \frac{1.1.3.5...2r-7}{2!(r-2)!} + \frac{1.3.1.3.5...2r-9}{3!(r-3)!} + \dots\right\}$$

$$= 4(-4)^{r-2}\left\{\frac{(\tfrac{1}{2})}{1!}\cdot\frac{(\tfrac{1}{2})(-\tfrac{1}{2})(-\tfrac{3}{2})...(-r-\tfrac{5}{2})}{(r-1)!} + \right.$$

$$\left. + \frac{(\tfrac{1}{2})(-\tfrac{1}{2})}{2!}\cdot\frac{(\tfrac{1}{2})(-\tfrac{1}{2})...(-r-\tfrac{7}{2})}{(r-2)!} + \dots\right\}$$

where the braces contain the coefficient of x^r in the expansion of $(1+x)^{1/2}(1+x)^{1/2}$, except for two terms $1\times\frac{(\tfrac{1}{2})(-\tfrac{1}{2})...(-r-\tfrac{3}{2})}{r!}$. For $r \geqslant 2$, the coefficient of x^r in $(1+x)$ is zero, so

$$D_{r+1} = 4(-4)^{r-2}(-2)\,\frac{(\tfrac{1}{2})(-\tfrac{1}{2})...(-r-\tfrac{3}{2})}{r!}$$

$$= 2^{r-1}\cdot\frac{1.3.5...2r-3}{r!} = \frac{(2r-2)!}{(r-1)!r!}\text{ , as required.}$$

If we are allowed to use *both* the above arguments, and hence both (1) and (2) the problem is much simplified, since multiplying (1) by $(n-1)$ and subtracting [(2), after writing $n-1$ for n,] we have

$$(n-1)D_n - 2(n-4)D_{n-1} = (n-1)(D_2D_{n-1} + D_{n-1}D_2)$$

$$\therefore\quad (n-1)D_n = 2(2n-5)D_{n-1}\,. \qquad (4)$$

Successive application of (4) now gives (3). To obtain (3) directly from (1) is not so easy, but may be done by finding the generating function for D_n:

$$f(x) = \sum_{r=2}^{\infty} D_r x^{r-1}.$$

It will be seen that this definition, and the following manipulations, are justified in case $|x| < 1/4$.

We now form the 'Cauchy product' of this series with itself, combining the terms as shown by the diagonals in figure 2

$$\{f(x)\}^2 = D_2D_2x^2 + (D_2D_3+D_3D_2)x^3 + (D_2D_4+D_3D_3+D_4D_2)x^4 + \ldots$$

$$= D_3x^2 + D_4x^3 + D_5x^4 + \ldots \text{ by (1)}$$

$$= f(x) - x.$$

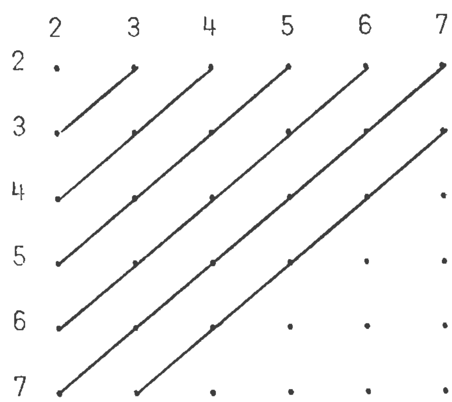

FIGURE 2

Solving this quadratic for $f(x)$, we have

$$f(x) = \tfrac{1}{2}\{1 - (1-4x)^{\frac{1}{2}}\} \qquad (5)$$

where the negative sign has been chosen to make $f(0) = 0$.

Then D_n is the coefficient of x^{n-1} in the binomial expansion of $-\frac{1}{2}(1-4x)^{\frac{1}{2}}$ $(n \geq 2)$, and is found to be given by (3).

D_n may be deduced in a similar way from (2), though a little calculus is required.

The simplicity of relation (4) suggests that it could be obtained by direct combinatorial argument from the original figure.

A clue might be that $2n-5$ is the total number of lines (sides and diagonals) in a dissection of a $(n-1)$-sided polygon. The factor 2 can always be introduced by giving these a direction, but this has not so far led to a soultion. Even more tantalising are the relationships

$$(2n-3)D_n = {}^{2n-3}C_{n-2} = {}^{2n-3}C_{n-1} \tag{6}$$

$$(n-1)D_n = {}^{2n-4}C_{n-2} \tag{7}$$

$$(n-3)D_n = 2.{}^{2n-5}C_{n-4} = 2.{}^{2n-5}C_{n-1} \tag{8}$$

$$(n-1)D_n = 2.{}^{2n-5}C_{n-3} = 2.{}^{2n-5}C_{n-2} \tag{9}$$

each of which suggests that there should be a very simple 'choice' argument, giving the required formula immediately. I have been unable to find such an argument.

3. A more difficult question than the original one is 'how many essentially different dissections are there?', in the following sense. If the polygon is regular, how many dissections are there which cannot be obtained from one another by rotation or reflexion? For example, the original problem was illustrated by 5 diagrams (figure 3) giving $D_5 = 5$. However, these become mere rotations of each other if the pentagon is regular. Again the 14 dissections of a hexagon are rotations or reflexions of one of the three types shown in figure 4.

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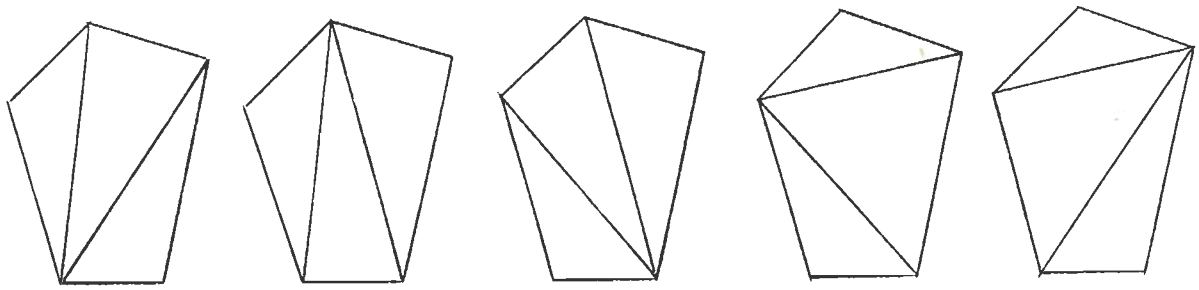

FIGURE 3

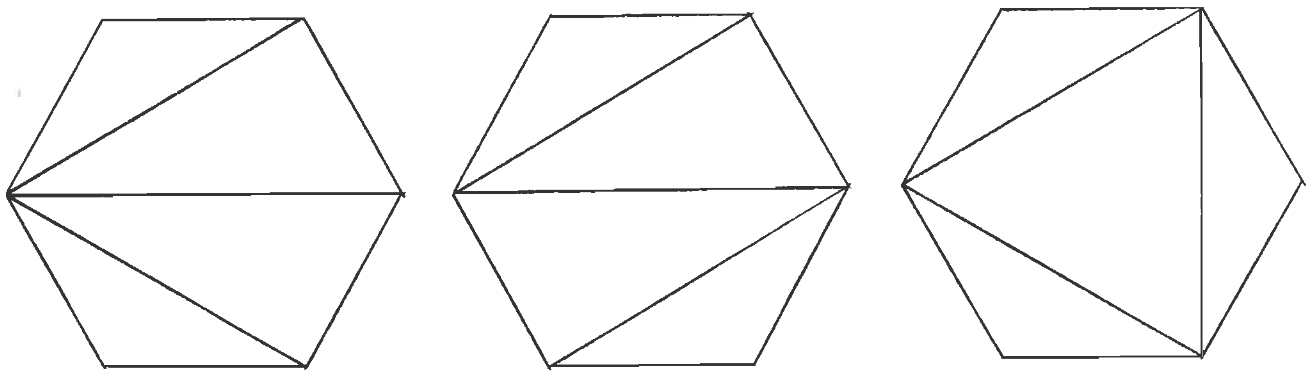

FIGURE 4

The dissections may be divided into 6 types according to their symmetries:

(a) reflexion in an axis only, having the symmetry of a kite. S_n

(b) rotation through 180° only, having the symmetry of a parallelogram. P_n

(c) rotation through 120° only. Possible only if *n* is a multiple of 3. T_n

(d) combination of (a) and (b), having the symmetries of a rectangle or rhombus. Only possible if *n* is a multiple of 4. R_n

(e) combination of (a) and (c), having the symmetries of an equilateral triangle. Only possible if *n* is a multiple of 6. Q_n

(f) unsymmetrical, having none of the above symmetries. U_n

We will denote the number of essentially different dissections of each of these types by S_n, P_n, T_n, R_n, Q_n and U_n respectively. Then the total number of essentially different dissections is

$$E_n = P_n + Q_n + R_n + S_n + T_n + U_n \qquad (10)$$

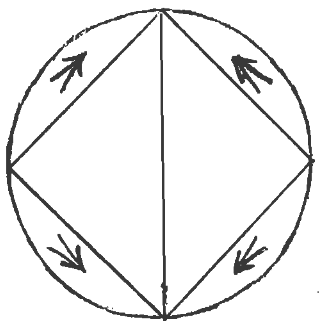

FIGURE 5

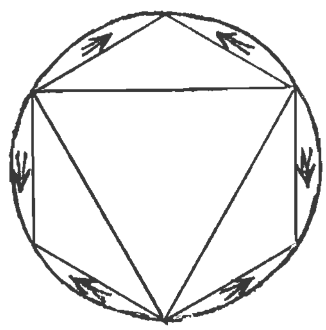

FIGURE 6

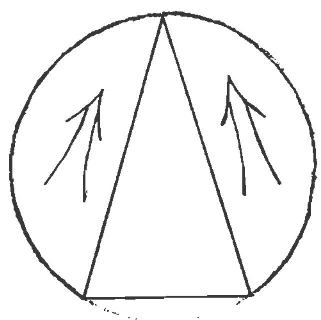

FIGURE 7

type 1

From figure 5, $R_n = D_{\frac{1}{2}n+1}$ (11)

If we define $D_n = 0$ when n is not an integer, then (11) is true for $n = 2,3,4,\ldots$, except that we may define $R_2 = 1$.

From figure 6, $Q_n = D_{\frac{1}{6}n+1}$ (12)

This is again true for all $n \geq 2$, except that $Q_3 = 1$.

We calculate S_n in two cases. If n is odd, there must be a triangle as shown in figure 7 and then the remainder can be filled in $D_{\frac{1}{2}(n+1)}$ ways. This will include the Q_n in the case where $n = 3$ only. Since n is odd, it can only vacuously include the R_n.

$S_n = D_{\frac{1}{2}(n+1)}$ if n odd

If n is even, the dissection will contain a kite as shown in figure 8 (see page 8), with one or other diagonal drawn. Fixing our attention on the symmetrical diagonal, we see that half the polygon can be dissected in $D_{\frac{1}{2}n+1}$ ways. These will

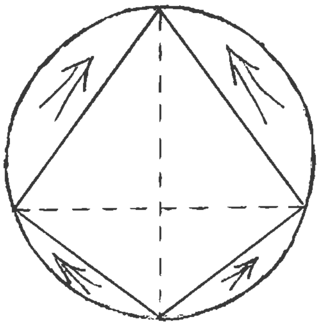

FIGURE 8

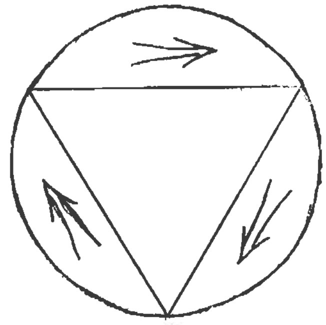

FIGURE 9

include the P_n in case n is a multiple of 4. The remainder occur in mirror image pairs, so we have $\frac{1}{2}(D_{\frac{1}{2}n+1} - R_n)$ distinct contributions to the S_n. The diagonal may be drawn in two ways, giving $D_{\frac{1}{2}n+1} - R_n$. These will also include the Q_n. Hence

$$S_n = D_k - R_n - Q_n, \tag{13}$$

where $k = \frac{1}{2}(n + 1)$ or $\frac{1}{2}n + 1$ according as n is odd or even.

The P_n will contain a parallelogram and one of its diagonals. Fixing this we see that

$$P_n = \frac{1}{2}(D_{\frac{1}{2}n+1} - R_n) \tag{14}$$

as in the above argument.

From figure 9, and as in preceding arguments

$$T_n = \frac{1}{2}(D_{\frac{1}{3}n+1} - Q_n) \tag{15}$$

Finally U_n may be calculated from the total number of dissections, since each of the P_n, Q_n, R_n, S_n, T_n, U_n, occur n, $\frac{1}{3}n$, $\frac{1}{2}n$, n, $\frac{2}{3}n$ and $2n$ times respectively. I.e.

$$D_n = nP_n + \frac{1}{3}nQ_n + \frac{1}{2}nR_n + nS_n + \frac{2}{3}nT_n + 2nU_n \tag{16}$$

Solution of (10) - (16) gives

$$E_n = \frac{1}{2n}D_n + \frac{1}{4}D_{\frac{1}{2}n+1} + \frac{1}{2}D_{\overline{\frac{1}{2}n+1}} + \frac{1}{3}D_{\frac{1}{3}n+1} \tag{17}$$

where the vinculum in the third term is to be removed in case n

is even. Here is a table of results for small values of n.

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n	D_n	P_n	Q_n	R_n	S_n	T_n	U_n	E_n
2	1	-	-	1	-	-	-	1
3	1	-	1	-	-	-	-	1
4	2	-	-	1	-	-	-	1
5	5	-	-	-	1	-	-	1
6	14	1	1	-	1	-	-	3
7	42	-	-	-	2	-	2	4
8	132	2	-	1	4	-	5	12
9	429	-	-	-	5	1	21	27
10	1430	7	-	-	14	-	61	82
11	4862	-	-	-	14	-	214	228
12	16796	20	1	2	39	2	669	733
13	58786	-	-	-	42	-	2240	2282
14	208012	66	-	-	132	-	7330	7528
15	742900	-	-	-	132	7	24695	24834
16	2674440	212	-	5	424	-	83257	83898
17	9694845	-	-	-	429	-	284928	285357 ✓
18	~~87638810~~ 35357670	715 ✓	2 ✓	- ✓	1428 ✓	20 ✓	~~1044444~~ 981079 ✓	~~1046609~~ 983244
19	129644790	-	-	-	1430	-	3410990	3412420
20	477638700	2424	-	14	4848	-	11937328	11944614
21	1767263190	-	-	-	4862	66	42075242	42080170
22	6564120420	8398	-	-	16796	-	149171958	149197152
23	24466267020	-	-	-	16796	-	531866972	531883768
24	91482563640	29372	5	42	58739	212	1905842605	1905930975
25	343059613650	-	-	-	58786	-	6861162880	6861221666
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I am grateful to W.G. Brown for permission to reproduce the references to his paper [54] and to him, to I.M.H. Etherington, J.W. Moon and L. Moser, for additional items.

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