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STATISTICAL MECHANICS AND PARTITIONS INTO
NON-INTEGRAL POWERS OF INTEGERS

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1. The problem of the partition of numbers, first investigated in detail by Hardy
and Ramanujan (1), has in recent years assumed importance on account of its application
by Bohr and Kallikar (2) in evaluating the density of energy levels in heavy nuclei.
A 'physical approach' to the partition theory has been made by Auluck and Kothari (3),
who have studied the properties of quantal statistical assemblies corresponding to the
partition functions familiar in the theory of numbers. The thermodynamical approach
to the partition theory, apart from its intrinsic interest, draws attention to aspects
and generalizations of the partition problem that would, otherwise, perhaps go un-
noticed. Thus we are led to consider restricted partitions such as: partitions where the
summands are repeated not more than a specified number of times; partitions where
the summands are all different; partitions into summands which must not be less than
a specified value; partitions into a prescribed number of summands, and so on. The
generalization that seemed to us to be the most interesting is the extension of the
partition concept to include partitions into non-integral powers of integers. The
problem is then to count the number of solutions satisfying the equation

\[ a_1^\sigma + a_2^\sigma + a_3^\sigma + \ldots \leq q, \]

where

\[ 1 \leq a_1 \leq a_2 \leq a_3 \ldots. \]

\( q \) is the integer to be partitioned and \( \sigma \) is a positive number not necessarily integral.
The number of solutions in the above case is denoted by \( c(q; \sigma) \). In the case where the
summands are all different, i.e.,

\[ 1 \leq a_1 < a_2 < a_3 \ldots, \]

the number of solutions is denoted by \( c^*(q; \sigma) \). The partition functions† \( p(q; \sigma) \) and
\( p^*(q; \sigma) \) which enumerate the partitions of \( q \) into \( \sigma \) th powers of integers are then given
by

\[
P(q; \sigma) = c(q; \sigma) - c(q-1; \sigma), \]

\[
P^*(q; \sigma) = c^*(q; \sigma) - c^*(q-1; \sigma). \]

(1)

The appropriate thermodynamic assembly corresponding to the above partition
functions is one obeying Bose-Einstein or Fermi-Dirac statistics and containing an
indefinite number of similar particles, the energy levels of any individual particle
being given by

\[ e_r = \sigma r^\sigma \quad (r = 1, 2, 3, \ldots), \]

† It will, of course, be noted that we use here the term partition function as it is used in the
theory of numbers. This is not to be confused with the use of the term in statistical mechanics,
where it signifies the sum taken suitably over the states of the assembly.
where $x$ is a constant of the dimensions of energy. In the present note, the assembly is restricted in the sense that the total number of particles is a fixed number $n$. The relevant partition functions in the Fermi-Dirac and Bose-Einstein cases are then $P^\sigma_n(q; \sigma)$ and $p_n(q; \sigma)$, where the latter enumerates the partitions of $q$ into $n$ different summands and the latter the partitions of $q$ into $n$ or less summands with repetitions allowed. We find

$$P^\sigma_n(q; \sigma) \sim \exp\left\{\pi \sqrt{\frac{3q}{2}} \frac{1}{\sigma^2} q^{1(1-\sigma)}\right\}$$

and

$$p_n(q; \sigma) \sim \exp\left\{(\sigma + 1) \left[\frac{1}{\sigma} \Gamma\left(1 + \frac{1}{\sigma}\right)\right]^{1(1-\sigma)} q^{1(1+\sigma)}\right\}.$$  

With the approximate method developed here one cannot obtain the multiplier of the exponential expressions for the partition functions correctly. $q_1$ in the formula for $P^\sigma_n(q; \sigma)$ denotes the excitation energy of the Fermi-Dirac assembly and is given by

$$q = q_1 + \frac{n^{1-\sigma}}{2}.$$ 

It is striking that for $\sigma = 1$ we obtain

$$P^\sigma_n(q) = p_n(q_1),$$

(2)

where

$$q = q_1 + \frac{n^2}{2}.$$ 

For $\sigma > 1$, $P^\sigma_n(q; \sigma) < p_n(q_1; \sigma)$ and for $\sigma < 1$, $P^\sigma_n(q; \sigma) > p_n(q_1; \sigma)$ and the two are equal only in the limiting case of $\sigma = 1$. The equality (2) is shown to hold in the analytical theory also. It is also noticeable that the exponential term in the expression for $p_n(q; \sigma)$ is the same as that for $p(q; \sigma)$. It has been shown that in the Bose case which is characterized by $n \gg q^2$ the function $p_n(q; \sigma)$ tends to $p(q; \sigma)$.

2. We proceed to evaluate the relevant thermodynamic functions. We consider a 'particle' the number of states of which lying between the energies $\epsilon$ and $\epsilon + d\epsilon$ is given by

$$a(\epsilon) d\epsilon = B e^{\epsilon - T} d\epsilon,$$

(3)

where $B$ and $\rho$ are constants and $\rho \gg 1$. The distribution law for an assembly composed of such 'particles' is given by

$$n(\epsilon) d\epsilon = \frac{a(\epsilon) d\epsilon}{\epsilon T + \beta},$$

where $T$ is the temperature (in energy units) of the assembly, $\beta = -1$ for Bose-Einstein statistics and $\beta = +1$ for Fermi-Dirac statistics. The parameter $A$ is independent of $\epsilon$ and is a function of the total number of particles in the assembly.

We shall first consider the Bose-Einstein degenerate case. For this case $A = 1$, and we have for the total energy of the assembly the expression

$$E = B \int^\infty_0 \frac{e^\epsilon d\epsilon}{e^\frac{\epsilon T}{T} - 1} = BT^{\rho+1} \Gamma(\rho + 1) \xi(\rho + 1).$$

(4)

The entropy $S$ of the assembly is given by

$$S = \frac{\rho + 1}{\rho} \frac{E}{T}.$$  

† The case of the unrestricted partitions is taken in § 4.
Eliminating $T$ between (4) and (5) we have

$$S = \frac{\rho + 1}{\rho} [B \Gamma (\rho + 1) \xi (\rho + 1)]^{1/(\rho + 1)} E^{(\rho + 1)}.$$  

(6)

Consider now the Fermi-Dirac degenerate case which is characterized by $A \gg 1$. The energy $E$ is given by

$$E = B T^{\rho - 1} \int_0^\infty \frac{u^\rho du}{A e^u + 1}.$$  

Making use of Sommerfeld's Lemma (4) according to which for large $A$

$$\int_0^\infty \frac{u^\rho du}{A e^u + 1} = \frac{\nu_0^{\rho + 1}}{\rho + 1} \left[ 1 + \frac{\pi^2 (\rho + 1)}{6 \nu_0^2} + \frac{7 \pi^4 (\rho + 1)^2 (\rho - 1)^2 (\rho - 2)^2}{360 \nu_0^4} + \ldots \right],$$

where $\nu_0 = \log A$ and the error is of the order of $1/A$, we have

$$E = B T^{\rho - 1} \frac{\nu_0^{\rho + 1}}{\rho + 1} \left[ 1 + \frac{\pi^2 (\rho + 1)}{6 \nu_0^2} + \ldots \right].$$  

(7)

For the total number of systems in the assembly we have

$$n = B T^\rho \frac{\nu_0^\rho}{\rho} \left[ 1 + \frac{\pi^2 (\rho - 1) \rho}{6 \nu_0^2} + \ldots \right].$$

From the above

$$\nu_0 = \log A = (\Gamma (\rho + 1) A_1)^{1/\rho} \left[ 1 - \frac{\pi^2}{6} (\rho - 1) \frac{\rho}{A_1} + \ldots \right],$$

where

$$A_1 = \frac{\rho^\rho}{B T^\rho \Gamma (\rho)}.$$  

(8)

Substituting for $\nu_0$ in (7) we have, after a little algebra,

$$E = \frac{\rho}{\rho + 1} n T (\Gamma (\rho + 1) A_1)^{1/\rho} \left[ 1 + \frac{\pi^2 (\rho + 1)}{6} (\Gamma (\rho + 1) A_1)^{2\rho} + \ldots \right].$$  

(9)

For the free energy $F$ we have

$$F = \frac{\rho n T}{\rho + 1} (\Gamma (\rho + 1) A_1)^{1/\rho} \left[ 1 - \frac{\pi^2}{6} (\rho + 1) \frac{1}{(\Gamma (\rho + 1) A_1)^{2\rho}} + \ldots \right],$$

so that the entropy $S$ is given by

$$S = \frac{E - F}{T} = \frac{\pi^2}{3} \frac{n \rho}{(\Gamma (\rho + 1) A_1)^{1/\rho}},$$

(10)

the higher order terms being negligible when we consider the case for large $n$. Substituting the value of $A_1$ from (8) in (10) we finally have

$$S = \frac{\pi^2}{3} \rho^{1/\rho} B T^{1/\rho} (\Gamma (\rho + 1) A_1)^{1/\rho}.$$  

(11)

The first term in (9) is independent of $T$ and hence represents the zero-point energy of the assembly. With $A_1$ given by (8) the zero-point energy is

$$E_0 = \frac{\rho}{\rho + 1} \frac{n^{1+1/\rho}}{B^{1/\rho}} \rho^{1/\rho}.$$  

(12)
The second term in (9) which gives the excitation energy reduces to

$$E_1 = \frac{\pi^2}{6} \rho \eta T = \frac{\pi^2}{6} \rho^{(2-\lambda)/\rho} B^{\lambda/\rho} T^2. \quad (13)$$

Eliminating $T$ between (11) and (13) we have

$$S = \pi \sqrt{\frac{3}{2} E_1} \eta^{(1-\lambda)/\rho} \rho^{(1-\lambda)/\rho} B^{\lambda/\rho}. \quad (14)$$

The expressions (6) and (14) for the entropy will be used later.

3. The connecting link between thermodynamics and partition theory is provided by Bose's theorem (5) which connects the number of accessible wave functions of an assembly in the energy state $E$ with the entropy $S$ of the assembly according to the relation

$$\omega(E) = \frac{e^S}{\left(-2\pi T^2 \frac{\partial E}{\partial T}\right)^{1/2}}. \quad (15)$$

The considerations outlined in § 2 can be adapted to the partition problem if we contemplate the energy levels of the individual particles of the assembly to be described by

$$\epsilon_r = \alpha r^\sigma \quad (r = 1, 2, 3, \ldots),$$

where $\sigma$ is a positive number, not necessarily integral, and $z$ is a constant of the dimensions of energy. We then have

$$a(\epsilon) d\epsilon = d\epsilon = \frac{\alpha^{-1/\sigma}}{\sigma} \epsilon^{(1-\sigma)/\sigma}, \quad (16)$$

and the total energy of the assembly is

$$E = \sum_r n(\epsilon_r) z^\sigma \quad (16')$$

or

$$E = q = \sum_r n(\epsilon_r) r^\sigma.$$

An integer $q$ has thus been partitioned in $\sigma$th powers of integers.

A comparison between equations (3) and (16) gives

$$\rho = \frac{1}{\sigma}, \quad B = \frac{\rho}{\alpha^\sigma}. \quad (17)$$

With these values of $\rho$ and $B$ equations (6) and (14) become

$$S = (\sigma + 1) \left[ \frac{1}{\sigma} \Gamma \left( 1 + \frac{1}{\sigma} \right) \xi \left( 1 + \frac{1}{\sigma} \right) \right]^{(\sigma+1)} q^{(\sigma+1)} \quad (B.E. \ case) \quad (18 -)$$

and

$$S = \sigma^{-1} n \frac{\Gamma(1-\sigma)^2 \pi(\sigma)}{2} \quad (F.D. \ case). \quad (18 +)$$

The relation (9) becomes

$$q = q_1 + \frac{n^{1+\sigma}}{1+\sigma}. \quad (19)$$

In a Bose-Einstein degenerate assembly the number of accessible wave functions is enumerated by the number of ways in which $q$ energy quanta can be distributed among $n$ systems, no restriction being placed on the number assigned to any system. This enumeration for the partitions (16') is identical with the function $p_n(q; \sigma)$ so that we have from (15) and (18 -)

$$p_n(q; \sigma) \sim \exp \left[ (\sigma + 1) \left[ \frac{1}{\sigma} \Gamma \left( 1 + \frac{1}{\sigma} \right) \xi \left( 1 + \frac{1}{\sigma} \right) \right]^{(\sigma+1)} q^{(\sigma+1)} \right]. \quad (20)$$

Therefore for $q \gg q_1$, we have

$$p(q; \sigma) \sim q^{(\sigma+1)}.$$ 

and therefore

$$\ln p(q; \sigma) \sim (\sigma+1) \ln q.$$
Also \( p_n(q; \sigma) = \sum_{r=1}^{n} P_r(q; \sigma) \).

In the accompanying graphs we have plotted the enumerated values of the function \( P_r(q; \sigma) \) for \( q = 10 \) and for values of \( r \) ranging from 1 to 10. Two values of \( \sigma \) have been chosen, namely, \( \sigma = \frac{1}{3} \) and \( \sigma = \frac{2}{3} \). It is evident that for \( r \approx q^\frac{1}{3} \) the function has its maximum value, and for values for \( r \) greater than \( q^\frac{1}{3} \) the function falls off very rapidly.

Therefore for \( r \gg q^\frac{1}{3} \), which condition is fulfilled in the Bose-Einstein degenerate case, we have
\[
\sum_{r=1}^{n} P_r(q; \sigma) \rightarrow \sum_{r=1}^{q} P_r(q; \sigma),
\]
and therefore
\[
p_n(q; \sigma) \rightarrow p(q; \sigma),
\]
so that
\[
p(q; \sigma) \sim \exp \left[ (\sigma + 1) \left( \frac{1}{\sigma} \Gamma \left( 1 + \frac{1}{\sigma} \right) \zeta \left( 1 + \frac{1}{\sigma} \right) \right) q^{(\sigma+1)} \right].
\]
This is identical with the Hardy-Ramanujan expression for \( p(q; \sigma) \). This case therefore leads to no new result. It must be emphasized that the method developed here is only approximate and cannot yield any multiplier of the exponential part of the partition function.

In Fermi-Dirac statistics we have to impose the restriction that the number of quanta attached to different particles must all be different. The number of accessible wave functions will then be equivalent to the partition function \( P^*_n(q; \sigma) \), so that (15) and (18) give

\[
P^*_n(q; \sigma) \sim \exp \{ \pi \sqrt{\frac{3}{2} q_1 n^{1-\sigma}/\sigma} \}.
\]

For \( \sigma = 1 \)

\[
P^*_n(q; \sigma) \sim \exp \{ \pi \sqrt{\frac{3}{2} q_1} \}.
\]

Also from (20)

\[
p_n(q_1) \sim \exp \{ \pi \sqrt{\frac{3}{2} q_1} \},
\]

so that we are led to the result

\[
P^*_n(q) = p_n(q_1),
\]

with

\[
q = q_1 + \frac{1}{2} n^2.
\]

To compare the relative magnitudes of the functions \( P^*_n(q; \sigma) \) and \( p_n(q_1; \sigma) \) for values of \( \sigma \) other than unity, we note that

\[
\log \frac{P^*_n(q_1; \sigma)}{p_n(q_1; \sigma)} \sim (\sigma + 1) \left( \frac{1}{1} \Gamma \left( 1 + \frac{1}{1} \right) \xi \left( 1 + \frac{1}{1} \right) \right) \left( \frac{q_1(\sigma + 1)}{\sigma} \right)^{\tau} - n \left( \frac{q_1 n^{1-\sigma}/\sigma}{\sigma} \right)^{\tau}.
\]

\( q_1 \) and \( n \) are large quantities and may be assumed to be of the same order of magnitude, i.e.

\[
q_1 \sim n = R.
\]

The relative magnitude of the two terms on the right in (24) is then given by

\[
\frac{c_1}{c_2} = \frac{(\sigma + 1) \left( \frac{1}{1} \Gamma \left( 1 + \frac{1}{1} \right) \xi \left( 1 + \frac{1}{1} \right) \right) q_1^{\tau} (\sigma + 1)}{\pi \left( \frac{q_1 n^{1-\sigma}/\sigma}{\sigma} \right)^{\tau}} \sim R^{\tau+1/2}. \quad (25)
\]

For \( \sigma > 1 \):

\[
\frac{c_1}{c_2} > 1 \quad \text{or} \quad p_n(q_1; \sigma) > P^*_n(q; \sigma).
\]

For \( \sigma < 1 \):

\[
\frac{c_1}{c_2} < 1 \quad \text{or} \quad p_n(q_1; \sigma) < P^*_n(q; \sigma).
\]

In the limiting case of

\[
\sigma = 1, \quad c_1 \sim c_2,
\]

and

\[
p_n(q_1) = P^*_n(q).
\]

This equality can be shown to hold good in the analytical theory also.

For this we note that \( P^*_n(q) \) is the coefficient of \( x^{2-1(n+1)} \) in

\[
\frac{1}{(1-x)(1-x^2)(1-x^3)\ldots(1-x^n)}.
\]

Also \( p_n(q_1) \) is the coefficient of \( x^{n} \) in

\[
\frac{1}{(1-x)(1-x^2)(1-x^3)\ldots(1-x^n)}.
\]

If

\[
P^*_n(q) = p_n(q_1),
\]

then the coefficient of \( x^{2-1(n+1)} \) in

\[
\frac{1}{(1-x)(1-x^2)(1-x^3)\ldots(1-x^n)}
\]

must be equal to the coefficient of \( q \) and \( \sigma \).

The conditions (23) and (24) above are satisfied if

\[
p_n(q_1; \sigma) = P^*_n(q; \sigma),
\]

where

\[
\frac{p_n(q_1; \sigma)}{P^*_n(q; \sigma)} \sim R^{\tau+1/2}.
\]

By following the method of proof used for the case of \( \sigma = 1 \) we can show that (20) and (24) hold for all \( \sigma \) greater than unity.

As an illustration, we quote the results for the cases of \( \sigma = \frac{1}{2}, \frac{3}{2} \), and \( \sigma = \frac{5}{2} \), respectively.

For \( \sigma = \frac{3}{2} \), the values of \( p_n(q_1; \sigma) \) are enumerated with the condition that

\[
1 \leq q_1 \leq q_2 \leq \ldots
\]

In Table 1(i) the values of \( q_1, q_2, \ldots \)

where

\[
C^*_n(q; \sigma) \text{ and the sum}
\]

of \( q \) and \( \sigma \).
must be equal to the coefficient of \( x^n \) in
\[
\frac{1}{(1-x)(1-x^2)(1-x^3)\ldots(1-x^{\alpha})},
\]
so that we should have the relation
\[
q - \frac{1}{2} n(n+1) = q_1.
\]
(25)

The conditions (23) and (25) are not materially different in the case we are considering, since all the formulae are asymptotic in character.

1. We now come to unrestricted partitions. First we note that for integral values Hardy and Ramanujan (1) obtained the asymptotic expression for \( p(q; \sigma) \):
\[
p(q; \sigma) = (2\pi)^{-\frac{1}{2}(\sigma+1)}\left(\frac{\sigma}{\sigma+1}\right)^{\frac{1}{2}} \beta q^{-\frac{1}{4}(\sigma+1)} \exp\left\{\left(\sigma + 1\right) \beta q^{\frac{1}{4}(\sigma+1)}\right\},
\]
(26)
where
\[
\beta = \left[\frac{1}{\sigma} \Gamma\left(1+\frac{1}{\sigma}\right) \zeta\left(1+\frac{1}{\sigma}\right)\right]^{\frac{1}{2}(\sigma+1)}.
\]
(27)

The corresponding expression for \( p^*(q; \sigma) \) has been given by Auluck and Kothari (6):
\[
p^*(q; \sigma) = \frac{1}{2\sqrt{\pi}} (1 - 2^{-\frac{1}{2} \sigma})^{\frac{1}{2}(\sigma+1)} \left(\frac{\beta \sigma}{\sigma+1}\right)^{\frac{1}{2}} q^{-\frac{1}{4}(2\sigma+1)(\sigma+1)}
\times \exp\left\{\left(\sigma + 1\right) \left(1 - \frac{1}{24\sigma}\right) \beta q^{\frac{1}{4}(\sigma+1)}\right\},
\]
(28)

By following the method outlined in the present paper it can readily be shown that even for the case of non-integral \( \sigma \) the exponential terms for \( p(q; \sigma) \) and \( p^*(q; \sigma) \) are identical with (26) and (28) respectively. As already remarked the method of the present paper is not competent to determine the multiplying factor to the exponential terms. However, the conjecture may be permissible that the asymptotic expressions (26) and (28) will hold for non-integral powers of \( \sigma \).

As an illustration we have carried out the enumeration of the partitions for two non-integral powers \( \sigma = \frac{1}{3} \) and \( \sigma = \frac{2}{5} \) for values of \( q \) up to 10. Table 1 gives the enumerated values of the function \( C_n(q; \sigma) \) which denotes the number of solutions satisfying
\[
a_1^\sigma + a_2^\sigma + \ldots + a_n^\sigma \leq q,
\]
where \( 1 \leq a_1 \leq a_2 \leq \ldots \leq a_n \) are any \( n \) positive integers.

In Table 1 (i) the number 15, for example, corresponding to \( q = 5 \) and \( n = 2 \) for the value \( \sigma = \frac{2}{5} \), represents all possible solutions of the equation
\[
a_1 + a_2^4 \leq 5,
\]
where
\[
1 \leq a_1 \leq a_2.
\]

If the summands are required to be all different the partitions are denoted by \( C_n^s(q; \sigma) \) and the enumerations in this case are tabulated in Table 2 for the same values of \( q \) and \( \sigma \).
Table 1

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Table 2

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In Table 2 (i) q ≤ \( \frac{3}{2} \), zero. Similarly
In Table 2(i) the partitions of $q$ for any values up to 10 into more than four parts is zero. Similarly in Table 2(ii) the maximum value of $n$ is 5.
For these enumerated values the functions $c(q; \sigma)$ and $c^*(q; \sigma)$ which as defined in §1 are given by

$$c(q; \sigma) = \sum_{r=1}^{\infty} C_r(q; \sigma)$$

and

$$c^*(q; \sigma) = \sum_{r=1}^{\infty} C^*_r(q; \sigma)$$

are easily found and are given in Table 3.

In the tables under the columns calculated we have given the values of the integrals

$$c(q; \sigma) = \int_1^q p(x; \sigma) \, dx,$$

$$c^*(q; \sigma) = \int_1^q p^*(x; \sigma) \, dx$$

(29)

evaluated numerically, the values of $p(q; \sigma)$ and $p^*(q; \sigma)$ being given by (26) and (28).

The agreement between the values calculated according to (29) and the values obtained by actual enumeration is satisfactory.†

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† It may also be noted that the agreement between the calculated and enumerated values of $c^*(q; \sigma)$ (Fermi-Dirac assemblies) is better than that in the case of $c(q; \sigma)$ (Bose-Einstein assemblies). In a recent paper Temperley (7) has shown that there is considerable doubt about the validity of the method of steepest descent when applied to Bose-Einstein assemblies, but no such difficulty seems to exist with Fermi-Dirac assemblies.

REFERENCES

See also Van Lier, C., and Uhlenbeck, G. E. Physica, 4 (1937), 531.

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