Machine methods of computation & numerical analysis, MIT, Quarterly Progress Report no. 17 Sept 15, 1955

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1173
273
→ 666
250
88
1174

$$\mathcal{P}\mathcal{F}(x) \equiv (\mathcal{P}[\mathcal{F}(x)], \mathcal{F}(x)), \quad x \geq 0.$$

$\mathcal{P}\mathcal{F}(x)$ is thus a random variable defined to the space of all pairs (ρ, g), $\rho \in \pi(g)$, $g \in \gamma$. $\mathcal{P}\mathcal{F}(x) = (\rho, g)$ is the event that at x $\mathcal{F}(x) = g$ and the process $\mathcal{F}$ arrived at g by passing through the sequence ρ. It is clear that for all $g \in \gamma$

$$P(\mathcal{F}(x) = g) = \sum_{\rho \in \pi(g)} P(\mathcal{P}\mathcal{F}(x) = (\rho, g)).$$

The theorem can now be stated. In the statement of the theorem "$*$" represents the operation of convolution.

Theorem: Suppose assumptions (1)-(7) hold. If for each $g \in \gamma$ $\alpha(G)$ = const. $= \alpha(g)$ for all $G \in g$, then $\mathcal{P}\mathcal{F}$ is Markovian and

$$P_2(g, x \mid g_0, 0) = \sum_{\rho \in \pi(g)} \left[\prod_{k=1}^{N} \frac{\bar{\pi}(g_{k-1} \to g_k)}{\alpha(g_{k-1})} \right] \alpha(g_0) e^{-x\alpha(g_0)} * \cdots * \frac{\alpha(g_N) e^{-x\alpha(g_N)}}{\alpha(g_N)},$$

$g_N = g$, where for each fixed $g' \in \gamma$, $\bar{\pi}(g' \to g)/\alpha(g')$ is a probability measure over the space γ.

The proof of this theorem will be contained in a paper yet to be published entitled, "The Concept of Enchainment--A Relation Between Stochastic Processes."

Bayard Rankin

References:

[1] Machine Methods of Computation and Numerical Analysis, Quarterly Progress Report No. 14, December 15 (1954), p. 45.

[2] ibid. Report No. 13, September 15 (1954), p. 48.

[3] ibid. Report No. 14, December 15 (1954), p. 11.

2.3 Final Reports

CALCULATION OF NUMBERS OF STRUCTURES OF RELATIONS ON FINITE SETS

A table of numbers of structures of dyadic relations has been calculated on Whirlwind I. The problem was taken up primarily to test a multi-register arithmetic program for manipulating numbers of arbitrary length. Thus, we obtained exact integer answers to this problem, even though these results are as high as 10^{60}. The results are given here completely written out, although they have primarily curiosity value.

The problem, as described in a previous report, [2] , concerns dyadic relationships holding among a set of n objects. A complete relationship is specified by an n x n matrix of 1's and 0's, a one in the ij place indicating that element i bears the relationship to element j while a zero indicates the absence of such a relationship. Counting the number of structures of relations amounts simply to counting the admissable arrays of 1's and 0's in the incidence matrix. With no further restrictions, we see that the answer is 2^{n^2}, but in this figure we have included many "orbits" of isomorphic structures which can be permuted into one another by renumbering the objects of the set. The task at hand is to find how many orbits of non-isomorphic structures exist. Davis [1] has shown that this number is

$$\text{(1)} \qquad str_n = \frac{1}{n!} \sum_{\tilde{\pi}} b(\pi)\ 2^{d(\pi)}$$

where the summand is to be evaluated for one permutation, $\tilde{\pi}$, from each conjugate class of the symmetric group of permutations on n objects. Every member of a conjugate class has the same distinct disjoint cycle scheme specified by

$$(p_1, p_2, \ldots, p_n)$$

where p_k is the number of cycles of length k in the permutation. The total number of conjugate classes is the number of partitions of n into integral summands. The quantity $b(\pi)$ is the redundancy, or number of member permutations in one conjugate class and is given by

$$b(\pi) = n!(1^{p_1}\ p_1!\ 2^{p_2}\ p_2! \ldots n^{p_n}\ p_n!)^{-1}$$

The quantity $d(\pi)$, known as the number of "degrees of freedom" connected with the permutation π, is defined by

$$d(\pi) = \sum_{h=1}^{n} \sum_{k=1}^{n} p_h \, p_k \, (h,k)$$

$$= 2 \sum_{h<k} p_h \, p_k \, (h,k) + \sum_{k=1}^{n} k \, p_k^2 .$$

(h,k) = greatest common divisor of h,k

Davis has developed other formulas for enumerating specialized classes of relation:

Non-isomorphic reflexive (or irreflexive) relations

$$ref_n = \frac{1}{n!} \sum_{\tilde{\pi}} b(\pi) \, 2^{d_{ref}(\pi)}$$

$$d_{ref}(\pi) = d(\pi) - \sum_{k=1}^{n} p_x$$

Non-isomorphic symmetric relations

$$sym_n = \frac{1}{n!} \sum_{\tilde{\pi}} b(\pi) \, 2^{d_{sym}(\pi)}$$

$$d_{sym}(\pi) = \sum_{k=1}^{n} p_k \left\{ \left[\frac{k}{2}\right] + 1 + k(p_k-1)/2 \right\}$$

$$+ \sum_{h<k} p_h \, p_k \, (h,k)$$

$\left[\frac{k}{2}\right]$= greatest integer function

Nonisomorphic irreflexive (or reflexive) symmetric relations

$$\text{irs}_n = \frac{1}{n!} \sum_{\tilde{\pi}} b(\pi)\, 2^{d_{irs}(\pi)}$$

$$d_{irs}(\pi) = d_{sym} - \sum_{k=1}^{n} p_k$$

Non-isomorphic anti-symmetric relations

$$\text{asym}_n = \frac{1}{n!} \sum_{\tilde{\pi}} b(\pi)\, 3^{d_{asym}(\pi)}$$

$$d_{asym}(\pi) = \sum_{k=1}^{n} p_k \left\{ \left[\frac{k-1}{2}\right] + k(p_k - 1)/2 \right\}$$

$$+ \sum_{h<k} p_h \cdot p_k \, (h,k)$$

Incidentally, note that ref_n is the number of directed graphs on n nodes and irs_n is the number of non-directed graphs.

All these formulas have been evaluated for n ranging up to 16 and the values are given in the accompanying tables.

Asymptotic Formulae - Inspection of the various enumeration formulae given above shows that the dominant contribution to the total number of structures is due to just one of the partitions. This partition is the one consisting of n 1-cycles and corresponds to the identity transform of the group of transforms of the incidence matrix. Taking this term from each of the formulas we have

$$\text{str}_n^2 \sim 2^{n^2}/n!$$

$$\text{ref}_n \sim 2^{n(n-1)}/n!$$

$$\text{sym}_n \sim 2^{(n+1)\frac{n}{2}}/n!$$

$$\text{irs}_n \sim 2^{\frac{n}{2}(n-1)}/n!$$

$$\text{asym}_n \sim 3^{\frac{n}{2}(n-1)}/n!$$

To show the accuracy of these approximations, we give Table VII as a representative table. It appears that the asymptotic formulae are good to about one per cent if the true structure number is of the order of 10^{10} and are (naturally) better for larger structure numbers.

M. Douglas McIlroy

References:

[1] R. L. Davis, Proc. Am. Math. Soc. 4(1953) 486

[2] M. D. McIlroy, Machine Methods of Computation and Numerical Analysis, Quarterly Progress Report No. 15 (1955) p. 10

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TABLE I Numbers of Structures of Relationships

n	all structures str_n	reflexive ref_n	symmetric sym_n	irreflexive symmetric irs_n	asymmetric $asym_n$
1	2	1	2	1	1
2	10	3	6	2	2
3	104	16	20	4	7
4	3044	218	90	11	42
5	$2.9197 \cdot 10^{5}$	9608	544	34	582
6	$9.6929 \cdot 10^{7}$	$1.5409 \cdot 10^{6}$	5096	156	21480
7	$1.1228 \cdot 10^{11}$	$8.8203 \cdot 10^{8}$	79264	1044	$2.1423 \cdot 10^{6}$
8	$4.5830 \cdot 10^{14}$	$1.7934 \cdot 10^{12}$	$2.2086 \cdot 10^{6}$	12346	$5.7502 \cdot 10^{8}$
9	$6.6666 \cdot 10^{18}$	$1.3028 \cdot 10^{16}$	$1.1374 \cdot 10^{8}$	$2.7467 \cdot 10^{5}$	$4.1594 \cdot 10^{11}$
10	$3.4939 \cdot 10^{23}$	$3.4126 \cdot 10^{20}$	$1.0926 \cdot 10^{10}$	$1.2005 \cdot 10^{7}$	$8.1601 \cdot 10^{14}$
11	$6.6603 \cdot 10^{28}$	$3.2523 \cdot 10^{25}$	$1.9564 \cdot 10^{12}$	$1.0190 \cdot 10^{9}$	$4.3744 \cdot 10^{18}$
12	$4.6557 \cdot 10^{34}$	$1.1367 \cdot 10^{31}$	$6.5234 \cdot 10^{14}$	$1.6509 \cdot 10^{11}$	$6.4540 \cdot 10^{22}$
13	$9.0169 \cdot 10^{40}$	$1.4669 \cdot 10^{37}$	$4.0540 \cdot 10^{17}$	$5.0502 \cdot 10^{13}$	$2.6378 \cdot 10^{27}$
14	$1.1521 \cdot 10^{48}$	$7.0316 \cdot 10^{43}$	$4.7057 \cdot 10^{20}$	$2.9054 \cdot 10^{16}$	$3.0037 \cdot 10^{32}$
15	$4.1233 \cdot 10^{55}$	$1.2583 \cdot 10^{51}$	$1.0231 \cdot 10^{24}$	$3.1426 \cdot 10^{19}$	$9.5773 \cdot 10^{37}$
16	$5.5343 \cdot 10^{63}$	$8.4446 \cdot 10^{58}$	$4.1788 \cdot 10^{27}$	$6.4001 \cdot 10^{22}$	$8.5888 \cdot 10^{43}$

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TABLE II Numbers of Structures of Dyadic Relations ÷2: A1173

n				str_n	÷ 2
1				2	1
2				10	5
3				104	52
4				3044	1522
5			2	91968	1 45984
6			969	28992	
			484	64496	
7		11	22829	08928	
		5	61414	54464	
8		45829	71000	61728	
		22914	85500	30864	
9	6666	62157	21539	27936	
	3333	31078	60769	63968	
10				3493	
				1746	
	90545	49349	98391	61856	
	95272	74674	99195	80928	
11			6660	34219	
	85078	18075	85386	36288	
12		46557	45648	25869	
	89066	03112	66511	04256	
13	901685	91267	11300	76041	
	19117	62528	96061	48096	
14			1152	05015	
	57604	74157	55389	34617	
	43236	77230	31424	28672	
15	4	12334	41401	68606	
	79295	18834	69376	48648	
	20973	59863	65854	35136	
16				5534	
	25727	62971	20722	05192	
	57533	09620	02145	19348	
	89642	93721	27245	80352	

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TABLE III Numbers of Structures of Reflexive (or irreflexive) Dyadic Relations

n				ref_n
1				1
2				3
3				16
4				218
5				9608
6			15	40944
7			8820	33440
8		179	33591	92848
9	13	02795	68243	99552
10	341260	43195	29725	80352
11			3	25229
	09385	05588	61111	97440
12		11	36674	54308
	25400	57443	38940	04224
13	146	69085	69271	29298
	69037	09607	53162	20928
14				7031
	56566	15234	99952	13855
	06555	97990	40912	17920
15		12	58345	26155
	04488	67281	04228	58105
	99188	12349	03206	83008
16	8444	60738	34225	80541
	87807	17815	32315	89171
	86915	03432	37883	67872

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TABLE IV Numbers of Structures of Symmetric Dyadic Relations

n					sym_n
1					2
2					6
3					20
4					90
5					544
6					5096
7					79264
8				22	08612
9				1137	43760
10			1	09262	27136
11			195	63634	35360
12			65233	50845	92096
13		405	40227	34209	96800
14	4	70568	64216	11199	63904
15	10230	63423	47118	94310	54720
16	41788492	03082	02323	60582	29792

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TABLE V Numbers of Structures of Irreflexive (or reflexive) Symmetric Dyadic Relations

n					irs_n
1					1
2					2
3					4
4					11
5					34
6					156
7					1044
8					12346
9				2	74668
10				120	05168
11				10189	97864
12			16	50911	72592
13			5050	20313	67952
14		29	05415	56572	35488
15		31426	48596	98043	08768
16	640	01015	70452	75578	94928

TABLE VI Numbers of Structures of Antisymmetric Dyadic Relations

n				$asym_n$
1				1
2				2
3				7
4				42
5				582
6				21480
7			21	42288
8			5750	16219
9		41	59392	43032
10		81600	74490	11040
11	4374	40620	99707	47314
12				645
	39836	93872	07497	39356
13			263	77967
	35571	22500	90533	73136
14		300	36589	61589
	80530	05349	84908	93399
15	957	72686	34898	11549
	49990	83757	92075	81003
16				8588
	84182	49161	16546	12893
	38402	27902	32471	44414

TABLE VII Comparison of Asymptotic Structure Formulae with True Formulae

	n = 7		n = 10		n = 15	
	approx. value	true value	approx. value	true value	approx. value	true value
str_n^2	$1.117\ 10^{11}$	$1.123\ 10^{11}$	$3.493\ 10^{23}$	$3.494\ 10^{23}$	$4.123\ 10^{55}$	$4.123\ 10^{55}$
ref_n			$3.411\ 10^{20}$	$3.413\ 10^{20}$	$1.258\ 10^{51}$	$1.258\ 10^{51}$
sym_n			$.993\ 10^{10}$	$1.093\ 10^{10}$	$1.016\ 10^{24}$	$1.023\ 10^{24}$
irs_n			$.970\ 10^{7}$	$1.201\ 10^{7}$	$3.102\ 10^{19}$	$3.143\ 10^{19}$
$asym_n$			$8.140\ 10^{14}$	$8.160\ 10^{14}$	$9.577\ 10^{37}$	$9.577\ 10^{37}$